

Deviation inequalities and CLT for random walks on acylindrically hyperbolic groups

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Abstract

We study random walks on groups with the feature that, roughly speaking, successive positions of the walk tend to be "aligned". We formalize and quantify this property by means of the notion of deviation inequalities. We show that deviation inequalities have several consequences including Central Limit Theorems, the local Lipschitz continuity of the rate of escape and entropy, as well as linear upper and lower bounds on the variance of the distance of the position of the walk from its initial point. In a second part of the paper, we show that the (exponential) deviation inequality holds for measures with exponential tail on acylindrically hyperbolic groups. These include non-elementary (relatively) hyperbolic groups, Mapping Class Groups, many groups acting on CAT(0) spaces and small cancellation groups.

Keywords and Phrases: random walks, rate of escape, entropy, Girsanov, hyperbolic groups.

1 Introduction

In this paper, we discuss fluctuations results for random walks on "hyperbolic-like" groups. In the sequel, G denotes an infinite, countable and discrete group with neutral element id equipped with a left-invariant metric d and we let μ denote a probability measure on G . We are interested in the behaviour of the random walk $(Z_n)_{n \geq 0}$ with driving measure μ and starting at $Z_0 = id$. The sequence $(Z_n)_{n \geq 0}$ is obtained as successive products of random variables $(X_j)_{j \geq 1}$ that are independent and with law μ . We let \mathbb{P}^μ be the law of the random walk and \mathbb{E}^μ be the corresponding expectation. Precise definitions and some background on random walks are given in sections 2 and 3.

We will work in the class of *acylindrically hyperbolic groups* as defined in [Osi14]. This is a large class of groups that includes non-elementary hyperbolic as well as relatively hyperbolic groups, Mapping Class Groups, many groups acting on CAT(0) spaces and small cancellation groups. We refer to sections 7 and 8 for definitions and references.

Our main result is a Central Limit Theorem for the distance of the walk to its starting point $d(id, Z_n)$, under the conditions explained below. A first step in the proof is to establish linear bounds on the variance. We also provide estimates of higher moments. Then we study fluctuations of the rate of escape and entropy in terms of the driving measure μ . We prove they are Lipschitz continuous and differentiable and we identify the derivative. All statements are given in the Conclusions at the end of the paper (Section 13).

Our assumptions on the random walk are expressed in terms of deviation inequalities. These measure how much successive positions of the walk fail to be "aligned". Thus, for a random walk

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satisfying a deviation inequality, given two integers n and m , with high probability, the distance $d(id, Z_{n+m})$ is close to the sum $d(id, Z_n) + d(Z_n, Z_{n+m})$. See Definition 3.2 and (5.20), (5.21). We show that the second moment deviation inequality is sufficient to get a Central Limit Theorem (Theorem 4.2).

Deviation inequalities are well-adapted to deal with random walks on groups with hyperbolic features. We prove deviation inequalities if G is hyperbolic and if μ satisfies some moment condition, Theorem 11.1. We also obtain deviation inequalities for acylindrically hyperbolic groups when μ has a finite exponential moment in Corollary 10.7.

Combining the results mentioned above, we deduce in particular the following:

Theorem 1.1. *Let G be an acylindrically hyperbolic group and let μ be a probability measure on G with a finite exponential moment and whose support generates G . Let d be a word metric on G . Then the law of $\frac{1}{\sqrt{n}}(d(id, Z_n) - \mathbb{E}^\mu[d(id, Z_n)])$ under \mathbb{P}^μ weakly converges to a Gaussian non-degenerate law.*

Theorem 1.1 actually applies to more general measures μ , as well as more general metrics d than stated in Theorem 1.1, see Theorem 13.4. The description of all metrics we can deal with involves the notion of acylindrically intermediate spaces, see Definition 10.1. For instance, they include the Teichmüller and Weil-Petersson metrics on Mapping Class Groups, see Proposition 10.2. Moreover, we recover the Central Limit Theorem for hyperbolic groups proven in [BQa] as a combination of the deviation inequality for hyperbolic group (Theorem 11.1) and Theorem 4.2.

Many results apply to other functionals than the distance to the initial point of the walk. We introduce the definition of “defective adapted cocycles”, see Definition 3.1 and state a C.L.T. in this context, Theorem 4.2. It includes the case of quasimorphisms, first obtained in [BH11].

Our theorems extend previously known results in two directions. On the one hand we deal with driving measures μ with an infinite support where most authors assume finite support or super-exponential moments. Most results are therefore new even for hyperbolic groups. On the other hand, we cover the case of acylindrically hyperbolic groups, which is vastly broader than that of hyperbolic groups, and general acylindrically hyperbolic groups do not even come with a compact boundary. Such extensions require a new approach and motivated us to develop the theory of deviation inequalities.

Context The history of Central Limit Theorems on groups with hyperbolic features can be traced back to early result of Furstenberg and Kesten in the sixties. We refer to [BQb] for historical background. The most recent contributions to this subject will be found in [BQa], where the C.L.T. is proved when G is hyperbolic and μ has a finite second moment, and in [Hor15], where a similar C.L.T. is proved for random walks on Mapping Class Groups and $Out(F_n)$.

The classical approach for the C.L.T., used in all the references we are aware of and whose most powerful version is in [BQa], assumes G has a nice action on some compact space X and derives the C.L.T. as a special case of a Central Limit Theorem for cocycles defined on X . When G is hyperbolic, X is the horo-functions (Busemann) boundary of (G, d) . In the case G is a Mapping Class Group, X is the boundary of Teichmüller space. As explained below, we take a radically different approach.

The question of the regularity of the rate of escape or the entropy in terms of the driving measure was first addressed in [Ers11]. Gouëzel recently proved in [Gou15] that, on a hyperbolic group G , the entropy and rate of escape are analytic functions on the set of probability measures μ with a given finite support. Once again the approach used in [Gou15], as well as in previous references like [Led12] and [Led13], uses some boundary theory. In particular one needs know the Martin boundary of the walk. (We recall that, when G is hyperbolic and μ is finitely supported, then [Anc90] and [Anc88] showed that the Martin boundary coincides with the Gromov boundary.)

Using compactifications to study random walks with a driving measure with a finite exponential moment on an acylindrically hyperbolic group seems hopeless. Indeed, first recall that on any non-elementary hyperbolic group one can construct a random walk whose driving measure has a finite

exponential moment but whose Martin boundary is not the Gromov boundary of the group, see [Gou13]. More dramatically, when G is only assumed to be acylindrically hyperbolic, we have to deal with actions of G on a hyperbolic spaces X that need not be locally compact.

There is a connection between the Central Limit Theorem and the regularity of the rate of escape. This is a rather general fact that has little to do with hyperbolicity. It appears in [Mat14] in the context of random walks on hyperbolic groups with a finitely supported driving measure and we shall exploit here too.

Let us mention that random walks on acylindrically hyperbolic group have been recently studied in [MT14] where it is shown that their Poisson boundary coincides with the Gromov boundary of X . This result was already known for hyperbolic groups, see [Kai00].

Deviation inequalities

Using deviation inequalities, we are able to completely avoid considering any compactification of our group or the space it acts on. This paper is 98 per cent self-contained. The main proofs rely on a combination of quite elementary probabilistic and geometric arguments (not much more than Markov's inequality and the triangle inequality indeed).

By definition, a random walk satisfies a deviation inequality in a given metric if the Gromov product between its initial point and two successive positions remains of order one in probability. Note that such a property cannot hold for almost all paths of the walk. (Typically one gets logarithmic divergences of the Gromov products.) We define different types of deviation inequalities depending on whether we require exponential or polynomial control over the tail of the law of the Gromov products, see Subsection 5.2 for this definition and Section 3 for the definition in the case of defective adapted cocycles.

Deviation inequalities imply some almost additivity properties: the law of large numbers can be made quantitative (Lemma 3.4). More is true: since the distance (or the cocycle) is almost additive along the trajectories of the walk, we deduce that the influence of a given increment is small. It then follows from an Efron-Stein type argument that the variance is sub-linear (Theorem 4.3) and almost additive (Theorem 4.1). The Central Limit Theorem itself follows from some strengthening of these additive properties: let us try to approximate the distance $d(id, Z_n)$ by a sum of independent random variables. The first layer of such an approximation would be to simply write the sum of the distances between successive positions of the walk. Doing so, we commit an error. The second layer is then to take into account corrections due to the Gromov products of successive triplets of successive positions, avoiding overlaps to maintain some independence. This is still not perfect but we get better approximations by considering further corrections corresponding to Gromov products involving positions that correspond to times that differ by two (still avoiding overlaps). Eventually, running this process a finite but large number of times leads to a decomposition of the distance $d(id, Z_n)$ as a sum of i.i.d. random contributions plus some error, see Subsection 4.2. This error is expressed in terms of Gromov products and, under the assumption of a deviation inequality, we show it has a small variance. Thus the proof of the C.L.T. boils down to (and only relies on!) the C.L.T. for i.i.d. random variables.

We now describe our strategy for the proof of deviation inequalities for random walks on acylindrically hyperbolic groups. Most arguments start with a group G acting on a (hyperbolic) metric space X and are in two steps:

1. [Probabilistic step] a sample path has with high probability a certain property P (this step usually does not use specific geometric properties of X).
2. [Geometric step] due to specific geometric properties of X and of the action of G , any path with property P is “close to being a geodesic” in the appropriate sense.

For example, (part of) property P can be that the distance between the i -th and j -th position, measured in the metric of X , is linear in $|i - j|$.

As a first step towards the proof of the deviation inequalities, we establish a linear progress property that says that, with overwhelming probability, the random walk tends to move away from its initial position at linear speed (Theorem 9.1). Note that this statement is proved in X and not in the group itself. (Observe that the conclusion of Theorem 9.1 is obviously true for random walks on non-amenable groups for a word metric. But since we need it in X , we have to face the fact that the distance in X may not be proper. Then hyperbolicity and acylindricity are important.)

We next deduce bounds on the probability that the distance between the position of the walk at some intermediate time k and a quasi-geodesic from the identity to the position at time n is larger than a given parameter (Theorem 10.6). The argument is: due to the hyperbolicity of X and due to the linear progress property, this distance can only be big if the walk performs very large jumps.

Following such a strategy we obtain the deviation inequality for random walks on acylindrically hyperbolic groups with a driving measure with exponential tail in Corollary 10.7. In the case G itself is hyperbolic, we obtain a sharper control on the deviations in terms of the tail of μ in Theorem 11.1.

Note that deviation inequalities were previously proved for hyperbolic groups and driving measures with finite support in [BHM11] as a consequence of quasi-conformal properties of the harmonic measure. We obtain them here in a completely different way.

The linear progress property was obtained by Maher and collaborators in [CM14, Theorem 5.35], [MT14, Theorem 1.2] for measures with bounded support (in the space being acted on). For our applications we need to deal with measures with exponential tail. We do not know whether the strategy of the aforementioned papers extends to this case.

The kind of control of the geometry of random walk paths we need to establish deviation inequalities for acylindrically hyperbolic groups is reminiscent of the estimates of tracking rates in [Sis14b].

Organization of the paper The first part of the paper introduces deviation inequalities and investigates their consequences. It is untitled "Using deviation inequalities".

In Section 2 we recall the definitions of the rate of escape and the entropy and provide some general references about random walks on groups. In Section 3, we fix our notation, define defective adapted cocycles and deviation inequalities and state the law of large numbers (Theorem 3.3 and Lemma 3.4). Section 4 is about the Central Limit Theorem for defective adapted cocycles satisfying deviation inequalities (Theorem 4.2): we first estimate the variance (Theorem 4.3) and prove it has a limit (Theorem 4.1) using a sub-additivity argument. The C.L.T. itself follows from an approximation of the distance $d(id, Z_n)$ by a sum of independent random variables, see Subsection 4.2. Subsection 4.3 contains estimates of higher moments (Theorem 4.8). In Subsection 4.4 we obtain lower bounds for the variance (Theorem 4.11) that ensure that our C.L.T. is non-degenerate.

In Section 5, we address the question of the regularity of the rate of escape in terms of the driving measure. We define a distance between probability measures with a fixed support in Subsection 5.1 and recall deviation inequalities. Subsection 5.3 contains some discussion of references. The main results are Theorem 5.1 about the Lipschitz regularity of the rate of escape and Theorem 5.2 about its differentiability. Similar issues are discussed for the entropy in Section 6. The main results are Theorem 6.1 about the Lipschitz regularity of the entropy and Theorem 6.2 about its differentiability. The connection between the entropy and a rate of escape is done using Green metrics that are recalled in Subsection 6.1. One needs some control on how the Green metric fluctuates with respect to μ and this is done in Proposition 6.3.

In the second part of the paper, untitled "Getting deviation inequalities", we establish deviation inequalities for different classes of random walks.

Section 7 starts with some references about acylindrically hyperbolic groups. The definition is given at the beginning of Section 8 along with a first consequence (Lemma 8.1). In Section 9, we prove

that random walks on acylindrically hyperbolic groups tend to escape from their initial position, see Theorem 9.1. In Section 10, we define acylindrically intermediate spaces (Definition 10.1) and provide examples (Proposition 10.2). The main result is Theorem 10.6 that gives a control on how much the trajectories of the random walk deviate from quasi-geodesics. An immediate consequence is the deviation inequality for random walks on acylindrically hyperbolic groups in Corollary 10.7. Section 11 is dedicated to deviation inequalities in hyperbolic groups, and the main result there is Theorem 11.1. In Section 12 we go back to acylindrically hyperbolic groups and study deviation inequalities in the Green metric (Theorem 12.1).

Finally, in Section 13 we collect all results about acylindrically hyperbolic groups that can be proven combining results in Part 3.3 with results in Part II.

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Part I

Using deviation inequalities

2 Motivation

Let G be an infinite, discrete group with neutral element id ; let μ be a probability measure on G . Let μ^n denote the n -th convolution power of μ .

If the support of μ generates an infinite semi-group, then the sequence μ^n converges to 0. One may measure the rate of decay of μ^n through the notion of entropy.

Let $H(\mu) := \sum_{x \in G} (-\log \mu(x)) \mu(x)$ be the entropy of μ and assume it is finite. Then the sequence $(H(\mu^n))_{n \in \mathbb{N}}$ is sub-additive and the following limit exists:

$$h(\mu) := \lim_{n \rightarrow \infty} \frac{1}{n} H(\mu^n). \quad (2.1)$$

The quantity $h(\mu)$ is called the **asymptotic entropy** of μ .

Another quantity of interest is the rate of escape: let d be a left-invariant metric on G . Assume that μ has a finite first moment in the metric d , namely that $\sum_{x \in G} d(id, x) \mu(x) < \infty$. Then the sequence $(\sum_{x \in G} d(id, x) \mu^n(x))_{n \in \mathbb{N}}$ is sub-additive. Therefore the limit

$$\ell(\mu; d) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{x \in G} d(id, x) \mu^n(x) \quad (2.2)$$

exists; it is called the **rate of escape** of μ in the metric d . Thus the rate of escape gives the mean distance to the identity of a random element of G sampled from the distribution μ^n .

We observed in [BHM08] that the entropy coincides with the rate of escape for a special choice of the distance d called the Green metric. Details on the Green metric are given in Section 6.

The notion of asymptotic entropy was introduced by A. Avez in [Ave72] in relation with random walk theory. In [Ave74], Avez proved that, whenever $h(\mu) = 0$, then μ satisfies the Liouville property: bounded, μ -harmonic functions are constant. The converse was proved later, see [Der80] and [KV83]. The Liouville property is equivalent to the triviality of the asymptotic σ -field of the random walk with

driving measure μ (its so-called Poisson boundary), see [Der80] and [KV83] again. In more general terms, the entropy plays a central role in the identification of the Poisson boundary of random walks in many examples. We refer in particular to [Kai00] for groups with hyperbolic features. In this latter case, the asymptotic entropy is also related to the geometry of the harmonic measure through a "dimension-rate of escape- entropy" formula, see [BHM11] and the references quoted therein.

The notion of rate of escape is also related to the potential theory on G . Assume that d is proper and that the support of μ generates the whole group. One shows, see [KL07], that if the probability measure μ has a finite first moment and is such that $\ell(\mu; d) > 0$ then at least one of the following two properties must hold: i) there exists an homomorphism from G to \mathbb{R} , say H , such that the image of μ through H has non zero mean; ii) the Poisson boundary is non trivial, i.e. there exist non constant bounded μ -harmonic functions on G .

In this paper, we shall be mostly concerned with non-amenable groups and assume that the support of μ generates G . In that case the Poisson boundary is never trivial.

Both the rate of escape and the asymptotic entropy have simple interpretations in terms of random walks: let $(Z_n)_{n \geq 0}$ be a random walk with driving measure μ . This means that the increments $X_n := Z_{n-1}^{-1}Z_n$ are independent random variables with common law μ . Kingman's sub-additive theorem implies that

$$\ell(\mu; d) = \lim_{n \rightarrow \infty} \frac{1}{n} d(id, Z_n) \quad (\text{respectively } h(\mu) = \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mu^n(Z_n)) , \quad (2.3)$$

where both limits hold for almost any path of the random walk.

In this paper, we are mostly concerned with fluctuations in the ergodic limits in (2.3), where the word "fluctuations" can be understood as "stochastic fluctuations with respect to the trajectories of the walk" or "fluctuations of the rate of escape or entropy" with respect to μ . Both types of fluctuations are actually closely related to each other, as we shall see.

3 Random walks, cocycles and deviation inequalities

3.1 Random walks

Consider the product space $\Omega := G^{\mathbb{N}^*}$ where $\mathbb{N}^* = \{1, \dots\}$. Let $(X_n)_{n \in \mathbb{N}^*}$ designate the coordinate maps from Ω to G : thus $X_n(\omega) = \omega_n$ for any sequence $\omega = (\omega_1, \dots) \in \Omega$ and $n \in \mathbb{N}^*$. We shall also consider the sequence of functions $(Z_n)_{n \in \mathbb{N}}$ recursively defined by $Z_0(\omega) = id$ and $Z_n(\omega) = Z_{n-1}(\omega)X_n(\omega)$ for $n \geq 1$. We think of a sequence $(Z_n(\omega))_{n \in \mathbb{N}}$ as describing a trajectory in the group G . Thus $Z_n(\omega)$ gives the position of the trajectory at time n , while $X_n(\omega) = (Z_{n-1}(\omega))^{-1}Z_n(\omega)$ gives its increment also at time n .

Following the usual usage in probability theory we often omit to indicate that random functions, as Z_n or X_n , depend on ω .

We equip Ω with the product σ -field (i.e. the smallest σ -field for which all functions X_n are measurable). We endow Ω with the product measure $\mathbb{P}^\mu := \mu^{\mathbb{N}^*}$. We use the notation \mathbb{E}^μ to denote the expectation with respect to \mathbb{P}^μ and \mathbb{V}^μ to designate the variance with respect to \mathbb{P}^μ .

Observe that the law of the sequence $(Z_n)_{n \in \mathbb{N}}$ under \mathbb{P}^μ is the law of a random walk driven by μ and started at id : its increments are independent and identically distributed random variables of law μ . In particular, for any $n \in \mathbb{N}$, the law of Z_n under \mathbb{P}^μ is the n -th convolution power of μ , μ^n .

Let \mathcal{F}_n denote the σ -field generated by the random variables X_1, \dots, X_n (i.e. the smallest σ -field on Ω for which all functions $(X_j)_{j \leq n}$ are measurable).

Let θ be the canonical shift on Ω and $\theta_n := \theta^n$. Observe that $Z_m \circ \theta_n = Z_n^{-1}Z_{n+m}$.

3.2 Defective adapted cocycles

Although we are primarily interested in studying the distance of a random walk from its initial point at large times, our main results hold in the more general case of "approximate cocycles" that we define below.

Definition 3.1. A defective adapted cocycle (D.A.C.) is a sequence of real-valued maps on Ω , say $\mathcal{Q} = (Q_n)_{n \in \mathbb{N}^*}$, where each map Q_n is measurable with respect to the σ -field \mathcal{F}_n . By convention we take Q_0 to be identically 0. The **defect** of \mathcal{Q} is the collection of maps $\Psi = (\Psi_{n,m})_{(n,m) \in \mathbb{N} \times \mathbb{N}}$ defined by

$$\Psi_{n,m}(\omega) = Q_{n+m}(\omega) - Q_n(\omega) - Q_m(\theta_n \omega).$$

Examples of defective adapted cocycles:

Here are the most important examples of D.A.C. to be considered in the sequel.

1. Let f be a function from G to \mathbb{R} and let $M_n(\omega) = \sum_{j=1}^n f(X_j(\omega))$. Then the sequence $\mathcal{M} := (M_n)_{n \in \mathbb{N}^*}$ is a defective adapted cocycle and its defect vanishes.

2. A special class of D.A.C. are those for which there exists a map q from Ω to \mathbb{R} such that $Q_n(\omega) = q(Z_n(\omega))$. We call them **end-point D.A.C.**

Define the differential $\partial q(g, h) := q(gh) - q(g) - q(h)$ and observe that

$$\Psi_{n,m} = \partial q(Z_n, Z_n^{-1} Z_{n+m}).$$

(This follows from the identity $Z_m \circ \theta_n = Z_n^{-1} Z_{n+m}$.)

When the function ∂q is uniformly bounded with respect to g and h , then the function q is called a **quasimorphism**.

Let d be a left-invariant metric on G (e.g. a word metric). Then the maps $Q_n(\omega) = d(id, Z_n(\omega))$ define an end-point D.A.C. Its defect is $\Psi_{n,m}(\omega) = -2(d(id, Z_{n+m}(\omega)))_{Z_n(\omega)}$ where

$$(x, y)_w := \frac{1}{2}(d(w, x) + d(w, y) - d(x, y))$$

is the **Gromov product** of points $x, y \in G$ with respect to the reference point $w \in G$ in the metric d . We call this D.A.C. the **length D.A.C.** (for the metric d).

3.3 Deviation inequalities

By "deviation inequality" we mean some control on how much a D.A.C. fails to be a true cocycle with respect to \mathbb{P}^μ . In the case of the length D.A.C., it will give a control on how much the successive positions of the random walk fail to follow a "straight line".

Definition 3.2. Let μ be a probability measure on G . Let $\mathcal{Q} = (Q_n)_{n \in \mathbb{N}^*}$ be a defective adapted cocycle with defect $\Psi = (\Psi_{n,m})_{(n,m) \in \mathbb{N} \times \mathbb{N}}$.

Let $p > 0$. We say that \mathcal{Q} satisfies the **p -th-moment deviation inequality** (with respect to the measure μ) if there exists a constant $\tau_p(\mathcal{Q}; \mu)$ such that for all n and m in \mathbb{N} then

$$\mathbb{E}^\mu[|\Psi_{n,m}|^p] \leq \tau_p(\mathcal{Q}; \mu). \quad (3.4)$$

We say that \mathcal{Q} satisfies the **exponential-tail deviation inequality** (with respect to the measure μ) if there exists a constant $\tau_0(\mathcal{Q}; \mu)$ such that for all n and m in \mathbb{N} and for all $c > 0$, then

$$\mathbb{P}^\mu[|\Psi_{n,m}| \geq c] \leq \tau_0(\mathcal{Q}; \mu)^{-1} e^{-\tau_0(\mathcal{Q}; \mu)c}. \quad (3.5)$$

Clearly the p -th-moment deviation inequality implies the p' -th-moment deviation inequality whenever $p \geq p'$ and the exponential-tail deviation inequality implies the p -th-moment deviation inequality for all $p > 0$.

Let $p > 0$. We say that \mathcal{Q} has **finite p -th moment** with respect to μ if $\mathbb{E}^\mu[|Q_1|^p] < \infty$. We say that \mathcal{Q} has **exponential tail** if there exists $\alpha > 0$ such that $\mathbb{E}^\mu[e^{\alpha|Q_1|}] < \infty$. We use the notation

$$\chi_p(\mathcal{Q}; \mu) := \mathbb{E}^\mu[|Q_1|^p].$$

3.4 Laws of large numbers for defective adapted cocycles

The following result is a consequence of a more general ergodic theorem proved by Y. Derriennic [Der83].

Theorem 3.3. *Let \mathcal{Q} be a defective adapted cocycle with a finite first moment and satisfying the first moment deviation inequality with respect to the probability measure μ . Then there exists a real number $\ell(\mathcal{Q}; \mu)$ such that the sequence $\frac{1}{n}Q_n$ converges to $\ell(\mathcal{Q}; \mu)$ in $L_1(\mathbb{P}^\mu)$.*

We call $\ell(\mathcal{Q}; \mu)$ the **rate of escape** of the D.A.C. \mathcal{Q} with respect to μ .

Proof. We apply Derriennic's result (Théorème 1 in [Der83]).

First we should check that all Q_n 's are integrable and that $\inf_n \frac{1}{n} \mathbb{E}^\mu[Q_n] > -\infty$.

We start from the identity $Q_{n+m} = Q_n + Q_m \circ \theta_n + \Psi_{n,m}$ and observe that $Q_m \circ \theta_n$ has the same law as Q_m . A simple induction argument based on the fact that $\mathbb{E}^\mu[|\Psi_{n,m}|] < \infty$ for all n and m shows that $\mathbb{E}^\mu[|Q_n|] < \infty$ for all n . It also proves that $\sup_n \frac{1}{n} \mathbb{E}^\mu[|Q_n|] < \infty$.

The other condition to be checked is that $\mathbb{E}^\mu[\Psi_{n,m}^+] \leq c_m$ for a sequence c_m such that $\frac{1}{m}c_m \rightarrow 0$. With our assumptions, one may take $c_m = \tau_1(\mathcal{Q}; \mu)$. □

Lemma 3.4. *Let \mathcal{Q} be a defective adapted cocycle with a finite first moment and satisfying the first moment deviation inequality with respect to the probability measure μ . Then, for all $n \geq 1$,*

$$|\frac{1}{n} \mathbb{E}^\mu[Q_n] - \ell(\mathcal{Q}; \mu)| \leq \frac{1}{n} \tau_1(\mathcal{Q}; \mu). \quad (3.6)$$

Proof. Taking the expectation in the identity $Q_{n+m} = Q_n + Q_m \circ \theta_n + \Psi_{n,m}$, we get that $|\mathbb{E}^\mu[Q_{n+m}] - \mathbb{E}^\mu[Q_n] - \mathbb{E}^\mu[Q_m]| \leq \tau_1(\mathcal{Q}; \mu)$. Therefore the two sequences $a_n := \tau_1(\mathcal{Q}; \mu) - \mathbb{E}^\mu[Q_n]$ and $b_n := \tau_1(\mathcal{Q}; \mu) + \mathbb{E}^\mu[Q_n]$ are both subadditive.

By Theorem 3.3, $\frac{1}{n}a_n$ converges to $-\ell(\mathcal{Q}; \mu)$. The subadditivity implies that $\frac{1}{n}a_n \geq -\ell(\mathcal{Q}; \mu)$ for all $n \geq 1$. Similarly, we get that $\frac{1}{n}b_n \geq \ell(\mathcal{Q}; \mu)$. In other words, $\frac{1}{n}(\tau_1(\mathcal{Q}; \mu) - \mathbb{E}^\mu[Q_n]) \geq -\ell(\mathcal{Q}; \mu)$ and $\frac{1}{n}(\tau_1(\mathcal{Q}; \mu) + \mathbb{E}^\mu[Q_n]) \geq \ell(\mathcal{Q}; \mu)$. These inequalities imply (3.6). □

4 Central limit theorems for defective adapted cocycles

Recall from section 3.4, that a D.A.C. with finite first moment and first-moment deviation inequality satisfies the law of large numbers with rate of escape

$$\ell(\mathcal{Q}; \mu) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}^\mu[Q_n].$$

Theorem 4.1. *Let \mathcal{Q} be a defective adapted cocycle. Assume that \mathcal{Q} has a finite second moment and satisfies the second-moment deviation inequality with respect to the probability measure μ . Then the variance $\frac{1}{n}\mathbb{V}^\mu(Q_n)$ has a limit as n tends to ∞ . We denote it by*

$$\sigma^2(\mathcal{Q}; \mu) := \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{V}^\mu[Q_n].$$

Theorem 4.2. *Let \mathcal{Q} be a defective adapted cocycle. Assume that \mathcal{Q} has a finite second moment and satisfies the second-moment deviation inequality with respect to the probability measure μ . Then the law of $\frac{1}{\sqrt{n}}(Q_n - \ell(\mathcal{Q}; \mu)n)$ under \mathbb{P}^μ weakly converges to the Gaussian law with zero mean and variance $\sigma^2(\mathcal{Q}; \mu)$.*

4.1 Existence of the variance of D.A.C.: proof of Theorem 4.1

We start with an a priori estimate on the variance of Q_n .

Theorem 4.3. *Let \mathcal{Q} be a defective adapted cocycle. Assume that \mathcal{Q} has a finite second moment and satisfies the second-moment deviation inequality with respect to the probability measure μ . Define $C^+(\mathcal{Q}; \mu) := 4\chi_2(\mathcal{Q}; \mu) + 16\tau_2(\mathcal{Q}; \mu)$. Then*

$$\mathbb{V}^\mu(Q_n) \leq C^+(\mathcal{Q}; \mu)n. \quad (4.7)$$

for all $n \geq 1$.

Below we give a proof of Theorem 4.3 using the Efron-Stein inequality. An alternative proof is given in Remark 4.7. The Efron-Stein inequality and some of its extensions are discussed in Section 4.3 and Remark 4.10.

The proof of Theorem 4.3 is based on the following **replacement trick**. Let $k \geq 1$ and let X'_k be a random variable with law μ and independent of the sequence $(X_j)_{j \geq 1}$. Let $X_j^{(k)}$ be the sequence obtained when replacing X_k by X'_k in the sequence $(X_j)_{j \geq 1}$. In other words $X_j^{(k)} = X_j$ for all $j \neq k$ and $X_k^{(k)} = X'_k$. We now denote with θ the shift operator operating on the two sequences $(X_j)_{j \geq 1}$ and $(X_j^{(k)})_{j \geq 1}$.

Recall that, for all $n \geq 1$, then Q_n is measurable with respect to \mathcal{F}_n . Therefore Q_n is of the form $Q_n = f(X_1, \dots, X_n)$ for some function f . We define $Q_n^{(k)} := f(X_1^{(k)}, \dots, X_n^{(k)})$. Note that $Q_n^{(k)} = Q_n$ for $n < k$. Also note that $Q_n^{(k)} \circ \theta_m = Q_n \circ \theta_m$ for all $m \geq k$.

Then $\mathcal{Q}^{(k)} := (Q_n^{(k)})_{n \geq 1}$ is a D.A.C. associated to the sequence of random variables $(X_j^{(k)})_{j \geq 1}$. We let $\Psi_{n,m}^{(k)} := Q_{n+m}^{(k)} - Q_n^{(k)} - Q_m^{(k)} \circ \theta_n$ be its defect.

For $n \geq k$, we have

$$\begin{aligned} Q_n^{(k)} &= Q_{k-1}^{(k)} + Q_{n-k+1}^{(k)} \circ \theta_{k-1} + \Psi_{k-1, n-k+1}^{(k)} \\ &= Q_{k-1}^{(k)} + Q_1^{(k)} \circ \theta_{k-1} + Q_{n-k}^{(k)} \circ \theta_k + \Psi_{1, n-k}^{(k)} \circ \theta_{k-1} + \Psi_{k-1, n-k+1}^{(k)}, \end{aligned}$$

and similarly

$$Q_n = Q_{k-1} + Q_1 \circ \theta_{k-1} + Q_{n-k} \circ \theta_k + \Psi_{1, n-k} \circ \theta_{k-1} + \Psi_{k-1, n-k+1}.$$

Since $Q_{k-1}^{(k)} = Q_{k-1}$ and $Q_{n-k}^{(k)} \circ \theta_k = Q_{n-k} \circ \theta_k$, we get that

$$Q_n^{(k)} - Q_n =$$

$$Q_1^{(k)} \circ \theta_{k-1} - Q_1 \circ \theta_{k-1} + \Psi_{1,n-k}^{(k)} \circ \theta_{k-1} - \Psi_{1,n-k} \circ \theta_{k-1} + \Psi_{k-1,n-k+1}^{(k)} - \Psi_{k-1,n-k+1}. \quad (4.8)$$

As a consequence

$$|Q_n^{(k)} - Q_n| \leq$$

$$|Q_1^{(k)} \circ \theta_{k-1}| + |Q_1 \circ \theta_{k-1}| + |\Psi_{1,n-k}^{(k)} \circ \theta_{k-1}| + |\Psi_{1,n-k} \circ \theta_{k-1}| + |\Psi_{k-1,n-k+1}^{(k)}| + |\Psi_{k-1,n-k+1}|. \quad (4.9)$$

Note that both terms $Q_1^{(k)} \circ \theta_{k-1}$ and $Q_1 \circ \theta_{k-1}$ have the same law as Q_1 . Similarly, $\Psi_{1,n-k}^{(k)} \circ \theta_{k-1}$ and $\Psi_{1,n-k} \circ \theta_{k-1}$ have the same law as $\Psi_{1,n-k}$, and $\Psi_{k-1,n-k+1}^{(k)}$ and $\Psi_{k-1,n-k+1}$ have the same law.

Proof of Theorem 4.3. Taking the expectation of the square in (4.9), we deduce that

$$\mathbb{E}^\mu[(Q_n^{(k)} - Q_n)^2] \leq (2\sqrt{\chi_2(\mathcal{Q}; \mu)} + 4\sqrt{\tau_2(\mathcal{Q}; \mu)})^2 \leq 8\chi_2(\mathcal{Q}; \mu) + 32\tau_2(\mathcal{Q}; \mu).$$

By the Efron-Stein inequality, see [Ste86], we have:

$$\begin{aligned} \mathbb{V}^\mu[Q_n] &\leq \frac{1}{2} \sum_{k=1}^n \mathbb{E}^\mu[(Q_n^{(k)} - Q_n)^2] \\ &\leq n(4\chi_2(\mathcal{Q}; \mu) + 16\tau_2(\mathcal{Q}; \mu)). \end{aligned}$$

□

Proof of Theorem 4.1.

We will use the following fact.

Lemma 4.4. [Ham62] *Let $(a_n)_{n=1,\dots}$ be a sequence of real numbers so that there exists $b \geq 0$ with the property that $a_{n+m} \leq a_n + a_m + b\sqrt{n+m}$ for each $m, n \geq 1$. Then $\frac{a_n}{n}$ converges to some $L < +\infty$.*

The starting point is the identity

$$Q_{n+m} = Q_n + Q_m \circ \theta_n + \Psi_{n,m}.$$

The two terms Q_n and $Q_m \circ \theta_n$ are independent and $Q_m \circ \theta_n$ has the same distribution as Q_m . Therefore

$$\mathbb{V}^\mu[Q_n + Q_m \circ \theta_n] = \mathbb{V}^\mu[Q_n] + \mathbb{V}^\mu[Q_m].$$

We can now apply the inequality

$$|\mathbb{V}^\mu(A + B) - \mathbb{V}^\mu(A)| \leq \mathbb{V}^\mu(B) + 2\sqrt{\mathbb{V}^\mu(A)\mathbb{V}^\mu(B)}$$

with $A = Q_n + Q_m \circ \theta_n$ and $B = \Psi_{n,m}$. Theorem 4.3 yields

$$\begin{aligned} |\mathbb{V}^\mu[Q_{n+m}] - \mathbb{V}^\mu[Q_n] - \mathbb{V}^\mu[Q_m]| &\leq \mathbb{V}^\mu[\Psi_{n,m}] + 2\sqrt{\mathbb{V}^\mu[Q_n] + \mathbb{V}^\mu[Q_m]} \sqrt{\mathbb{V}^\mu[\Psi_{n,m}]} \\ &\leq \tau_2(\mathcal{Q}; \mu) + 2\sqrt{C^+(\mathcal{Q}; \mu)(n+m)} \sqrt{\tau_2(\mathcal{Q}; \mu)}, \end{aligned}$$

since $\mathbb{V}^\mu[\Psi_{n,m}] \leq \mathbb{E}^\mu[\Psi_{n,m}^2]$.

Lemma 4.4 implies that $\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{V}^\mu[Q_n]$ exists and is finite, as required. □

4.2 A C.L.T. for D.A.C.: proof of Theorem 4.2

We split the trajectory of the walk until time n into successive blocks of length 2^M and express Q_n as a sum of the contributions of the different blocks, plus some cross terms. The cross terms are expressed in terms of defects and can be controlled by the deviation inequality. Thus we conclude that Q_n can be approximated by a sum of i.i.d. random variables and therefore that its law is close to Gaussian.

Lemma 4.5. *Let \mathcal{Q} be a defective adapted cocycle. Assume that \mathcal{Q} has a finite second moment and satisfies the second-moment deviation inequality with respect to the probability measure μ . Then*

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{n} \mathbb{V}^\mu [Q_n - \sum_{j=0}^{n2^{-M}-1} Q_{2^M} \circ \theta_{j2^M}] = 0. \quad (4.10)$$

Proof of (4.10).

First consider an integer n of the form 2^k . Iterating the identity

$$Q_{2n} = Q_n + Q_n \circ \theta_n + \Psi_{n,n},$$

we get that

$$Q_{2^k} = \sum_{j=0}^{2^k-1} Q_1 \circ \theta_j + \sum_{i=1}^k \sum_{j=0}^{2^{k-i}-1} \Psi_{2^{i-1}, 2^{i-1}} \circ \theta_{j2^i}.$$

Let us now take n of the form $n = 2^k + 2^l$, with $k < l$. We use the identity $Q_n = Q_{2^l} + Q_{2^k} \circ \theta_{2^l} + \Psi_{2^l, 2^k}$ to get that

$$Q_n = \Psi_{2^l, 2^k} + \sum_{j=0}^{n-1} Q_1 \circ \theta_j + \sum_{i=1}^l \sum_{j=0}^{n2^{-i}-1} \Psi_{2^{i-1}, 2^{i-1}} \circ \theta_{j2^i}.$$

More generally, let $n \geq 1$ and write a dyadic decomposition of n in the form

$$n = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \dots + \varepsilon_m 2^m$$

where the ε_j are either 0 or 1 and m is such that $\varepsilon_m = 1$. (In other words, m is the integer part of $\log_2 n$.) Then

$$Q_n = \sum_{j=0}^m \gamma_j + \sum_{j=0}^{n-1} Q_1 \circ \theta_j + \sum_{i=1}^m \sum_{j=0}^{n2^{-i}-1} \Psi_{2^{i-1}, 2^{i-1}} \circ \theta_{j2^i}. \quad (4.11)$$

In this last expression, all the terms γ_j are either 0 or of the form $\Psi_{a,b}$ for some values of a and b that can be computed in terms of n but whose value does not play any role in the sequel.

We note that in the expression (4.11), the terms $Q_1 \circ \theta_j$ corresponding to different values of j are independent and identically distributed. Likewise, the terms $\Psi_{2^{i-1}, 2^{i-1}} \circ \theta_{j2^i}$ corresponding to different values of j are also independent and identically distributed.

Choose $M \geq 1$ such that $M \leq m$.

We group the different terms in the sum $\sum_{j=0}^{n-1} Q_1 \circ \theta_j$ in packs of length 2^M to get that

$$\sum_{j=0}^{n-1} Q_1 \circ \theta_j = \sum_{j=0}^{n2^{-M}-1} \left(\sum_{t=0}^{2^M-1} Q_1 \circ \theta_t \right) \circ \theta_{j2^M} + R_M^{(0)}, \quad (4.12)$$

where $R_M^{(0)}$ is a sum of at most $2^M - 1$ terms of the form $Q_1 \circ \theta_a$ for some index a . Likewise, for any index $i \leq M$, let us write

$$\sum_{j=0}^{n2^{-i}-1} \Psi_{2^{i-1}, 2^{i-1}} \circ \theta_{j2^i} = \sum_{j=0}^{n2^{-M}-1} \left(\sum_{t=0}^{2^{M-i}-1} \Psi_{2^{i-1}, 2^{i-1}} \circ \theta_{t2^i} \right) \circ \theta_{j2^M} + R_M^{(i)}, \quad (4.13)$$

where $R_M^{(i)}$ is a sum of at most $2^M - 1$ terms of the form $\Psi_{b,b} \circ \theta_a$ for some values of a and b . Recall the expression

$$Q_{2^M} = \sum_{t=0}^{2^M-1} Q_1 \circ \theta_t + \sum_{i=1}^M \sum_{t=0}^{2^{M-i}-1} \Psi_{2^{i-1}, 2^{i-1}} \circ \theta_{t2^i}.$$

Summing (4.12) and (4.13), we get

$$\sum_{j=0}^{n-1} Q_1 \circ \theta_j + \sum_{i=1}^M \sum_{j=0}^{n2^{-i}-1} \Psi_{2^{i-1}, 2^{i-1}} \circ \theta_{j2^i} = \sum_{j=0}^{n2^{-M}-1} Q_{2^M} \circ \theta_{j2^M} + R_M, \quad (4.14)$$

where R_M is the sum of the $R_M^{(i)}$ for $i = 0 \dots M$.

From (4.11) and (4.14), we get the following decomposition

$$Q_n = \sum_{j=0}^m \gamma_j + \sum_{j=0}^{n2^{-M}-1} Q_{2^M} \circ \theta_{j2^M} + R_M + \sum_{i=M+1}^m \sum_{j=0}^{n2^{-i}-1} \Psi_{2^{i-1}, 2^{i-1}} \circ \theta_{j2^i}. \quad (4.15)$$

Let us now bound the variance of these terms. We use the notation $\tau_2 = \tau_2(\mathcal{Q}; \mu)$ and $\chi_2 = \chi_2(\mathcal{Q}; \mu)$.

We have $m \leq \log_2 n$. Since the variance of each γ_j is bounded by τ_2 , we have

$$\mathbb{V}^\mu \left[\sum_{j=0}^m \gamma_j \right] \leq (\log_2 n)^2 \tau_2. \quad (4.16)$$

Since there are at most 2^M terms of the form $Q_1 \circ \theta_a$ and at most $M2^M$ terms of the form $\Psi_{b,b} \circ \theta_a$ in R_M , we get that

$$\mathbb{V}^\mu[R_M] \leq 2^{2M+1} (\chi_2 + M^2 \tau_2). \quad (4.17)$$

We observe that, for a fixed i , the random variables $\Psi_{2^{i-1}, 2^{i-1}} \circ \theta_{j2^i}$ are i.i.d. Therefore the variance of $\sum_{j=0}^{n2^{-i}-1} \Psi_{2^{i-1}, 2^{i-1}} \circ \theta_{j2^i}$ is bounded by $n2^{-i} \tau_2$ and

$$\mathbb{V}^\mu \left[\sum_{i=M+1}^m \sum_{j=0}^{n2^{-i}-1} \Psi_{2^{i-1}, 2^{i-1}} \circ \theta_{j2^i} \right] \leq n \tau_2 \left(\sum_{i=M+1}^m 2^{-i/2} \right)^2 \leq n \tau_2 \left(\sum_{i=M+1}^{\infty} 2^{-i/2} \right)^2. \quad (4.18)$$

Using the inequality $\mathbb{V}[A+B+C] \leq (\sqrt{\mathbb{V}[A]} + \sqrt{\mathbb{V}[B]} + \sqrt{\mathbb{V}[C]})^2$, we deduce from (4.15), (4.16), (4.17) and (4.18) that

$$\frac{1}{n} \mathbb{V}^\mu \left[Q_n - \sum_{j=0}^{n2^{-M}-1} Q_{2^M} \circ \theta_{j2^M} \right] \leq \frac{1}{n} \left((\log_2 n \sqrt{\tau_2} + 2^{M+1} \sqrt{\chi_2 + M^2 \tau_2} + \sqrt{n \tau_2} \left(\sum_{i=M+1}^{\infty} 2^{-i/2} \right) \right)^2.$$

Therefore

$$\limsup_n \frac{1}{n} \mathbb{V}^\mu[Q_n - \sum_{j=0}^{n2^{-M}-1} Q_{2^N} \circ \theta_{j2^M}] \leq \tau_2 \left(\sum_{i=M+1}^{\infty} 2^{-i/2} \right)^2.$$

and we obtain (4.10) by letting M tend to ∞ . \square

We are now in the position to apply the following

Lemma 4.6. *Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of centered and square integrable random variables such that $\mathbb{V}[A_n]$ has a limit, say σ^2 . We assume that for all M , there exist square integrable random variables $A_n^{(M)}$ and $B_n^{(M)}$ such that $A_n = A_n^{(M)} + B_n^{(M)}$ and*
(i) $\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{V}[B_n^{(M)}] = 0$, (ii) for each M , there exists σ_M^2 such that $\mathbb{V}[A_n^{(M)}] \rightarrow \sigma_M^2$ and $A_n^{(M)} - \mathbb{E}[A_n^{(M)}]$ converges in distribution towards the Gaussian law with mean 0 and variance σ_M^2 . Then $\lim_M \sigma_M^2 = \sigma^2$ and A_n converges in distribution as n tends to ∞ towards the Gaussian law with mean 0 and variance σ^2 .

Proof of Lemma 4.6.

We denote with $\mathcal{N}(0, \sigma^2)$ the Gaussian law with 0 mean and variance σ^2 .

From the inequality

$$\mathbb{V}[A + B] \leq (\sqrt{\mathbb{V}[A]} + \sqrt{\mathbb{V}[B]})^2$$

one easily deduces that σ_M^2 converges to σ^2 .

Let g be a smooth, compactly supported function on \mathbb{R} .

Since A_n is centered, we must have $\mathbb{E}[B_n^{(M)}] = -\mathbb{E}[A_n^{(M)}]$. The function g is Lipschitz continuous; it follows that there exists a constant C such that

$$\begin{aligned} |\mathbb{E}[g(A_n)] - \mathbb{E}[g(A_n^{(M)} - \mathbb{E}[A_n^{(M)}])]| &\leq C \mathbb{E}[|A_n - A_n^{(M)} + \mathbb{E}[A_n^{(M)}]|] \\ &= C \mathbb{E}[|B_n^{(M)} - \mathbb{E}[B_n^{(M)}]|] \leq C \sqrt{\mathbb{V}[B_n^{(M)}]}. \end{aligned}$$

Applying conditions (i) and (ii), we get that

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} |\mathbb{E}[g(A_n)] - \int g d\mathcal{N}(0, \sigma_M^2)| = 0.$$

Since $\mathcal{N}(0, \sigma_M^2)$ weakly converges to $\mathcal{N}(0, \sigma^2)$, we are done. \square

End of the proof of Theorem 4.2.

We apply Lemma 4.6 to the random variables

$$A_n = \frac{1}{\sqrt{n}}(Q_n - \mathbb{E}^\mu[Q_n]) \text{ and } A_n^{(M)} = \frac{1}{\sqrt{n}} \sum_{j=0}^{n2^{-M}-1} Q_{2^N} \circ \theta_{j2^M} - \frac{1}{\sqrt{n}} \mathbb{E}^\mu[Q_n].$$

The claim (4.10) gives condition (i) in the Lemma.

Fix M . Then the random variables $(Q_{2^N} \circ \theta_{j2^M})_{j=0 \dots n2^{-M}-1}$ are square integrable, independent and identically distributed. Therefore the convergence of the variance and the Central Limit Theorem for $A_n^{(M)}$ are nothing but the C.L.T. for sums of i.i.d. variables.

We conclude that the distribution of $\frac{1}{\sqrt{n}}(Q_n - \mathbb{E}^\mu[Q_n])$ converges to a Gaussian law. Lemma 3.4 allows to replace $\mathbb{E}^\mu[Q_n]$ by $\ell(Q; \mu)$. \square

Remark 4.7. One may obtain a linear upper bound on the variance of Q_n (as in the statement of Theorem 4.3) using the decomposition (4.11).

We already observed that, in the expression (4.11), the terms $Q_1 \circ \theta_j$ corresponding to different values of j are independent and identically distributed. Likewise, the terms $\Psi_{2^{i-1}, 2^i-1} \circ \theta_{j2^i}$ corresponding to different values of j are also independent and identically distributed. Therefore, taking the variance in (4.11), we get that

$$\sqrt{\mathbb{V}^\mu[Q_n]} \leq m\sqrt{\tau_2(\mathcal{Q} : \mu)} + \sqrt{n\chi_2(\mathcal{Q} : \mu)} + \sqrt{\tau_2(\mathcal{Q} : \mu)n} \sum_{i \geq 1} 2^{-i/2}.$$

Remember that $m \leq \log_2(n)$. Thus the above inequality yields a linear upper bound on the variance $\mathbb{V}^\mu[Q_n]$ and we have an alternative proof of Theorem 4.3 that avoids using the Efron-Stein inequality.

4.3 Higher moments

In Theorem 4.3 we showed that a finite second moment and the second moment deviation inequality imply a linear upper bound on the variance of Q_n . The next Theorem exploits Burkholder's inequality to generalize this fact to other moments.

Theorem 4.8. For all $p > 1$, there exists a constant $c(p)$ such that for any defective adapted cocycle \mathcal{Q} that has a finite p -moment and satisfies the p -th moment deviation inequality with respect to the probability measure μ then

$$\mathbb{E}^\mu[|Q_n - \mathbb{E}^\mu[Q_n]|^p] \leq c(p)(\chi_p(\mathcal{Q}; \mu) + \tau_p(\mathcal{Q}; \mu)) n^{p/2}.$$

The proof uses Burkholder's inequality that we first recall:

Lemma 4.9. Let $p > 1$. There exists a constant $c_B(p)$ such that for any $n \geq 1$ and any martingale difference sequence $(Y_j)_{j=1, \dots, n}$ with finite p -th moment then

$$\mathbb{E}[|\sum_{j=1}^n Y_j|^p] \leq c_B(p) \mathbb{E}[(\sum_{j=1}^n Y_j^2)^{p/2}].$$

Remark 4.10. The argument below in particular works with $p = 2$. Then one may choose $c_B(2) = 1$ and the inequality in Lemma 4.9 becomes an equality. We thus recover Theorem 4.3 although with a different constant.

Proof of Theorem 4.8.

Recall that \mathcal{F}_j is the σ -field generated by the variables X_1, \dots, X_j . Eventually, we shall apply Burkholder's inequality to the sequence of conditional expectations $Y_j = \mathbb{E}^\mu[Q_n | \mathcal{F}_j] - \mathbb{E}^\mu[Q_n | \mathcal{F}_{j-1}]$.

Note that the sequence (Y_j) is indeed a martingale difference sequence in L_p . And also note that $\sum_{j=1}^n Y_j = Q_n - \mathbb{E}^\mu[Q_n]$.

Let us play the same replacement trick as in the proof of Theorem 4.3: we see that $Y_j = \mathbb{E}^\mu[Q_n - Q_n^{(j)} | \mathcal{F}_j]$. (Keep in mind that $Q^{(j)}$ is obtained by replacing X_j by an independent copy X'_j . Then X'_j is in particular independent of \mathcal{F}_j ; so that $\mathbb{E}^\mu[Q_n^{(j)} | \mathcal{F}_j] = \mathbb{E}^\mu[Q_n | \mathcal{F}_{j-1}]$.)

It follows from Hölder's inequality that $\mathbb{E}^\mu[|Y_j|^p] \leq \mathbb{E}^\mu[|Q_n - Q_n^{(j)}|^p]$ and then from (4.8) that

$$\mathbb{E}^\mu[|Y_j|^p] \leq 6^{p-1}(2\chi_p(\mathcal{Q}; \mu) + 4\tau_p(\mathcal{Q}; \mu)). \quad (4.19)$$

By Burkholder's inequality, we have

$$\begin{aligned}
\mathbb{E}^\mu[|Q_n - \mathbb{E}^\mu[Q_n]|^p] &= \mathbb{E}^\mu\left[\left|\sum_{j=1}^n Y_j\right|^p\right] \\
&\leq c_B(p) \mathbb{E}^\mu\left[\left(\sum_{j=1}^n Y_j^2\right)^{p/2}\right] \\
&\leq c_B(p) n^{p/2-1} \sum_{j=1}^n \mathbb{E}^\mu[|Y_j|^p] \\
&\leq c_B(p) n 6^{p-1} (2\chi_p(\mathcal{Q}; \mu) + 4\tau_p(\mathcal{Q}; \mu)).
\end{aligned}$$

□

4.4 Lower bound on the variance of end-point D.A.C.

The C.L.T. in Theorem 4.2 applies even in examples where the limiting variance $\sigma^2(\mathcal{Q}; \mu)$ vanishes.

We now give a linear *lower* bound on the variance of Q_n , when it is an end-point D.A.C. Note that the next Theorem in particular applies to the length D.A.C.

Theorem 4.11. *Let \mathcal{Q} be an end-point defective adapted cocycle. Assume that \mathcal{Q} has a finite second moment and satisfies the second-moment deviation inequality with respect to the probability measure μ . Assume that $\ell(\mathcal{Q}; \mu) > 0$. Then $\sigma^2(\mathcal{Q}; \mu) > 0$.*

Remark 4.12. *In the special case of quasimorphisms, the authors of [BH11] give a necessary and sufficient condition for the non-vanishing of the variance that is more precise than the one in Theorem 4.11.*

Proof of Theorem 4.11. The existence of the limit of $\frac{1}{n} \mathbb{V}^\mu[Q_n]$ is guaranteed by Theorem 4.1, hence it suffices to show $\limsup \frac{1}{n} \mathbb{V}^\mu[Q_n] > 0$.

For convenience, we will assume $\mu(id) \neq 0$, which we can arrange by passing to a convolution power μ^K so that $\mu^K(id) \neq 0$. In fact, $\limsup (\mathbb{V}^\mu[Q_n]/n) \geq \limsup (\mathbb{V}^{\mu^K}[Q_n]/(nK))$.

Let $\tilde{\mu}(a) = \mu(a)/(1 - \mu(id))$ for $a \neq id$ and $\tilde{\mu}(id) = 0$.

Define N_n to be the random variable $\#\{j \leq n : X_j = id\}$ that counts the number of null increments up to time n . Let $S_n = \inf\{m : m - N_m \geq n\}$ be the first time $m - N_m$ exceeds n . We set $\tilde{Z}_n := Z_{S_n}$.

The idea of the proof is to exploit the fluctuations of N_n .

Claim: Under \mathbb{P}^μ , the sequence (\tilde{Z}_n) is a random walk driven by $\tilde{\mu}$. Also, the two sequences (\tilde{Z}_n) and (N_n) are independent.

Proof. First observe that $\tilde{\mu}$ is the law of X_1 conditioned on the event $(X_1 \neq id)$.

Let M be an integer and n_1, \dots, n_M be integers. Note that if the event $\mathcal{A} := (N_j = n_j \forall j \leq M)$ is not empty, then there is a unique set $\mathbb{N} \subset \{1, \dots, M\}$ such that, on \mathcal{A} and for $j \leq M$, $X_j = id$ if and only if $j \in \mathbb{N}$. Conversely, once we know for which indices $j \leq M$ we have $X_j = id$, then we know the value of $N_j; j \leq M$. Therefore conditioning on \mathcal{A} is equivalent to conditioning on the event $(X_j = id \text{ iff } j \in \mathbb{N})$.

Under the conditional law given \mathcal{A} , the random variables $(X_j : j \notin \mathbb{N})$ are i.i.d. with law $\tilde{\mu}$.

Finally observe that the increments of the sequence $(\tilde{Z}_k : k \leq n_M)$ are the variables $(X_j : j \notin \mathbb{N})$.

So we conclude that, conditionally on \mathcal{A} , the increments of the sequence $(\tilde{Z}_k : k \leq n_M)$ are i.i.d. with law $\tilde{\mu}$. Thus

$$\mathbb{P}^\mu[\tilde{Z}_1 = z_1, \dots, \tilde{Z}_{n_M} = z_{n_M} ; N_1 = n_1, \dots, N_M = n_M]$$

$$= \mathbb{P}^{\tilde{\mu}}[Z_1 = z_1, \dots, Z_{n_M} = z_{n_M}] \mathbb{P}^{\mu}[N_1 = n_1, \dots, N_M = n_M]$$

for all choices of M , n_1, \dots, n_M and z_1, \dots, z_{n_M} . We deduce that, for all k , all z_1, \dots, z_k and n_1, \dots, n_k and any $M \geq k$ then

$$\begin{aligned} & \mathbb{P}^{\mu}[\tilde{Z}_1 = z_1, \dots, \tilde{Z}_k = z_k; N_1 = n_1, \dots, N_k = n_k; N_M \geq k] \\ &= \mathbb{P}^{\tilde{\mu}}[Z_1 = z_1, \dots, Z_k = z_k] \mathbb{P}^{\mu}[N_1 = n_1, \dots, N_k = n_k; N_M \geq k]. \end{aligned}$$

When M tends to ∞ then N_M converges to $+\infty$ in probability. Therefore, letting M tend to ∞ in the preceding equality, we get that

$$\begin{aligned} & \mathbb{P}^{\mu}[\tilde{Z}_1 = z_1, \dots, \tilde{Z}_k = z_k; N_1 = n_1, \dots, N_k = n_k] \\ &= \mathbb{P}^{\tilde{\mu}}[Z_1 = z_1, \dots, Z_k = z_k] \mathbb{P}^{\mu}[N_1 = n_1, \dots, N_k = n_k]. \end{aligned}$$

It indeed shows that, under \mathbb{P}^{μ} , the sequence (\tilde{Z}_n) is a random walk driven by $\tilde{\mu}$ and that the two sequences (\tilde{Z}_n) and (N_n) are independent. \square

A consequence of the claim is that for all $k \leq n$, the law of Z_n given $N_n = k$ is the law of \tilde{Z}_{n-k} . To see this, note that, on the event $N_n = k$, we have $S_{n-k} = n$ and therefore $\tilde{Z}_{n-k} = Z_n$. Therefore

$$\mathbb{P}^{\mu}[Z_n = z; N_n = k] = \mathbb{P}^{\mu}[\tilde{Z}_{n-k} = z; N_n = k] = \mathbb{P}^{\mu}[\tilde{Z}_{n-k} = z] \mathbb{P}^{\mu}[N_n = k].$$

We used the independence of \tilde{Z}_{n-k} and N_n .

Recall that an end-point D.A.C. is of the form $Q_n = q(Z_n)$ for some function q . Let us define $\tilde{Q}_n := q(\tilde{Z}_n)$. Then the sequence $\tilde{Q} := (\tilde{Q}_n)_{n \in \mathbb{N}^*}$ defines an end-point D.A.F. with respect to the random walk $(\tilde{Z}_n)_{n \in \mathbb{N}}$.

Let $\ell = \ell(Q; \mu)$. We will now show that there exists $\epsilon > 0$ so that for each sufficiently large n we have

$$\mathbb{E}^{\mu}[(Q_n - \ell n)^2] + \mathbb{E}^{\mu}[(Q_{n+r(n)} - \ell n - \ell r(n))^2] > \epsilon n,$$

where $r(n) = \lceil \sqrt{n} \rceil$.

The inequality above suffices to show that $\limsup \frac{1}{n} \mathbb{V}^{\mu}[Q_n] > 0$ in view of Lemma 3.4 which guarantees that ℓn is a good approximation of $\mathbb{E}^{\mu}[Q_n]$.

Observe that for each m we have

$$\begin{aligned} & \mathbb{E}^{\mu}[(Q_m - \ell m)^2] = \\ & \sum_{k \in \mathbb{N}} \mathbb{E}^{\mu}[(Q_m - \ell m)^2 | N_m = k] \mathbb{P}[N_m = k] = \sum_k \mathbb{E}^{\mu}[(\tilde{Q}_{m-k} - \ell m)^2] \mathbb{P}[N_m = k]. \end{aligned}$$

We can now use the fact that N_m has the same law as a sum of independent Bernoulli random variables with parameter $p = \mu(id)$, so that there exists $\epsilon_0 > 0$ so that for each large enough n and integer x satisfying $|x - pn| \leq 3\sqrt{n}$ we have $\mathbb{P}^{\mu}[N_n = x] \geq \epsilon_0/\sqrt{n}$.

Hence, for each sufficiently large n we have

$$\begin{aligned} & \mathbb{E}^{\mu}[(Q_n - \ell n)^2] + \mathbb{E}^{\mu}[(Q_{n+r(n)} - \ell n - \ell r(n))^2] \\ & \geq \frac{\epsilon_0}{\sqrt{n}} \left(\sum_{|k-pn| \leq r(n)} \mathbb{E}^{\mu}[(\tilde{Q}_{n-k} - \ell n)^2] + \sum_{|j-pr(n)-pn| \leq 3r(n)} \mathbb{E}^{\mu}[(\tilde{Q}_{n+r(n)-j} - \ell n - \ell r(n))^2] \right) \\ & \geq \frac{\epsilon_0}{\sqrt{n}} \left(\sum_{|k-pn| \leq r(n)} \mathbb{E}^{\mu}[(\tilde{Q}_{n-k} - \ell n)^2] + \sum_{|k-pn| \leq r(n)} \mathbb{E}^{\mu}[(\tilde{Q}_{n-k} - \ell n - \ell r(n))^2] \right). \end{aligned}$$

For any given k and any $A \in \mathbb{R}$, we have $(A - \ell n)^2 + (A - \ell n - \ell r(n))^2 \geq \ell^2 r(n)^2 / 2$, so that we get

$$\mathbb{E}^\mu[(Q_n - \ell n)^2] + \mathbb{E}^\mu[(Q_{n+r(n)} - \ell n - \ell r(n))^2] \geq \frac{\epsilon_0}{\sqrt{n}} 2r(n) \frac{\ell^2 r(n)^2}{2} \geq \epsilon_0 \ell^2 r(n)^2,$$

as required. \square

5 Fluctuations of the rate of escape

Let d be a left-invariant metric on G . In this section of the paper, we discuss regularity properties of the rate of escape $\ell(\mu; d)$ (as defined in (2.2) and (2.3)), considered as a function of the driving measure μ . In Theorem 5.1, we give sufficient conditions that imply the Lipschitz continuity of ℓ ; Theorem 5.2 is about the differentiability of ℓ .

5.1 Distances between measures

In the sequel we shall study the regularity of the rate of escape of probability measures with a fixed support. Let B be a (finite or infinite) subset of G . Let $\mathcal{P}(B)$ be the set of probability measures with support equal to B . We shall endow $\mathcal{P}(B)$ with the topology that we now describe.

Let μ_0 and μ_1 belong $\mathcal{P}(B)$ and let $\nu(\mu_0, \mu_1) := \sup_{a \in B} (\max(\frac{\mu_0(a)}{\mu_1(a)}; \frac{\mu_1(a)}{\mu_0(a)}) - 1)$. It is not difficult to see that ν defines a distance on $\mathcal{P}(B)$.

Assume that B is finite. We may identify $\mathcal{P}(B)$ as a subset of \mathbb{R}^d with $d = \#B$. Observe that $\nu(\mu_0, \mu_1)$ is then locally equivalent to the Euclidean distance between μ_0 and μ_1 .

We do not assume that B is finite any more. Let $\mu \in \mathcal{P}(B)$. By neighborhood of μ , we mean a set of the form $\mathcal{N} = \{\mu_0 \in \mathcal{P}(B); \nu(\mu_0, \mu) \leq K\}$ for some $0 < K$. Note that, for μ_0 and μ_1 in \mathcal{N} , then $\nu(\mu_0, \mu_1)$ is equivalent to the norm $\sup_{a \in B} |\mu_1(a) - \mu_0(a)| / \mu(a)$.

In the sequel, we will say that a function F is Lipschitz continuous on \mathcal{N} if it satisfies $|F(\mu_1) - F(\mu_0)| \leq C\nu(\mu_0, \mu_1)$ for some constant C and all μ_0 and μ_1 in \mathcal{N} .

5.2 Deviation inequalities

Let μ be a probability measure on G . We specialize the definition of deviations inequalities from Definition 3.2 in the case of the length D.A.C. Recall the definition

$$(x, y)_w := \frac{1}{2}(d(w, x) + d(w, y) - d(x, y))$$

for the **Gromov product** of points $x, y \in G$ with respect to the reference point $w \in G$ (in the metric d).

Thus we say that μ satisfies the **p -th-moment deviation inequality** (in the metric d) if there exists a constant $\tau_p(\mu)$ such that for all n and m in \mathbb{N} then

$$\mathbb{E}^\mu[(id, Z_{n+m})_{Z_n}^p] \leq \tau_p(\mu). \quad (5.20)$$

We say that μ satisfies the **exponential-tail deviation inequality** (in the metric d) if there exists a constant $\tau_0(\mu)$ such that for all n and m in \mathbb{N} and for all $c > 0$, then

$$\mathbb{P}^\mu[(id, Z_{n+m})_{Z_n} \geq c] \leq \tau_0(\mu)^{-1} e^{-\tau_0(\mu)c}. \quad (5.21)$$

We shall also use uniform versions of the deviation inequalities. Namely: let μ be a probability measure on G with support B .

We say that μ satisfies the **locally uniform p -th-moment deviation inequality** if there exists a neighborhood of μ in $\mathcal{P}(B)$, say \mathcal{N} , such that inequality (5.20) is satisfied by all measures in \mathcal{N} with the same constant. Similarly, we say that μ satisfies the **locally uniform exponential-tail deviation inequality** if there exists a neighborhood of μ in $\mathcal{P}(B)$, say \mathcal{N} , such that inequality (5.21) is satisfied by all measures in \mathcal{N} with the same constant.

Let $p > 0$. We say that μ has **finite p -th moment** with respect to d if $\sum_{x \in G} d(id, x)^p \mu(x) < \infty$. We say that μ has **exponential tail** if there exists $\alpha > 0$ such that $\sum_{x \in G} e^{\alpha d(id, x)} \mu(x) < \infty$. We use the notation

$$\chi_p(\mu; d) := \sum_{x \in G} d(id, x)^p \mu(x).$$

5.3 Lipschitz continuity and differentiability of the rate of escape

The question of the regularity of the rate of escape and the entropy as a function of μ was raised by A. Erschler and V. Kaimanovich in [EK13]. We refer to [GL13] for a review on the subject. Although the question is simple enough to state, very little is known for general (non-hyperbolic) groups. Only in a handful of examples, can we explicitly compute $\ell(\mu; d)$. In [EK13], it is proved that, for non-elementary hyperbolic groups and under a first moment assumption, then the asymptotic entropy is continuous for the weak topology on measures - a fact that fails to be true in all groups, see [Ers11]. If we restrict ourselves to measures μ with fixed finite support (and still assume that G is non-elementary hyperbolic), F. Ledrappier proved in [Led13] that h and ℓ are Lipschitz continuous. This result was upgraded to analyticity by S. Gouezel in a very recent preprint [Gou15]. We refer to [GL13] for a review of the state of the art before [Gou15].

Different techniques were used to prove these results. In [EK13], the authors use a version of Kaimanovich's ray criteria. The results of [Led12], [Led13] and [Gou15] are based on properties of the dynamics induced by a random walk on the boundary of G . [HMM13] proved the analyticity of the rate of escape for random walks in Fuchsian groups using their automatic structure and regeneration times.

In [Mat14], we introduced a martingale approach to clarify the connection between the differentiability of ℓ and h and the Central Limit Theorem. The proofs of both Theorem 5.1 and 5.2 follow a similar martingale approach.

Theorem 5.1. *Let μ be a probability measure on G with support B . Assume that μ has a finite first moment. Assume that μ satisfies the locally uniform first-moment deviation inequality. Then there exists a neighborhood of μ in $\mathcal{P}(B)$, say \mathcal{N} , such that the function $\mu \rightarrow \ell(\mu; d)$ is Lipschitz continuous on \mathcal{N} .*

Theorem 5.2. *Let μ be a probability measure on G with support B . Assume that μ has a finite second moment and satisfies the second-moment deviation inequality and the locally uniform first-moment deviation inequality. Then the function $\mu_0 \rightarrow \ell(\mu_0; d)$ is differentiable at $\mu_0 = \mu$ in the following sense: Let $(\mu_t, t \in [0, 1])$ be a curve in $\mathcal{P}(B)$ such that $\mu_0 = \mu$ and, for all $a \in B$, the function $t \rightarrow \log \mu_t(a)$ has a derivative at $t = 0$, say $\nu(a)$. We assume that ν is bounded on B and also that $\sup_{t \in [0, 1]} \sup_{a \in B} |\frac{1}{t} \log \frac{\mu_t(a)}{\mu_0(a)} - \nu(a)| < \infty$. Then the limit of $\frac{1}{t}(\ell(\mu_t; d) - \ell(\mu; d))$ as t tends to 0 exists. Besides this limit coincides with the covariance*

$$\sigma(\nu, \mu; d) := \lim_n \frac{1}{n} \mathbb{E}^\mu[d(id, Z_n) \left(\sum_{j=1}^n \nu(X_j) \right)]. \quad (5.22)$$

Observe that $\sigma(\nu, \mu; d)$ is linear w.r.t. ν .

The proofs of both Theorems are based on the Girsanov formula that we recall below.

Theorem 5.1 directly follows from the Girsanov formula, the replacement trick from Section 4.1 and the deviation inequality. The strategy to obtain Theorem 5.2 is the same as in [Mat14]. It very much relies on the expression of the derivative of the variance at a fixed time as a correlation as in (5.28), and the Central Limit Theorem 4.2.

5.4 Proof of Theorem 5.1

For $t \in [0, 1]$ and $a \in G$, we define $\mu_t(a) := \mu_0(a) + t(\mu_1(a) - \mu_0(a))$. Note that μ_t is a probability measure in $\mathcal{P}(B)$ for all $t \in [0, 1]$.

We define $\nu_0(a) := (\mu_1(a) - \mu_0(a))/\mu_0(a)$ for $a \in B$ and more generally $\nu_t(a) := (\mu_1(a) - \mu_0(a))/\mu_t(a)$ for $a \in B$ and $t \in [0, 1]$. Observe that $\nu(\mu_0, \mu_1) := \sup_{a \in B} \sup_{t \in [0, 1]} |\nu_t(a)|$. (The \sup_t is actually a max and is attained at either $t = 0$ or $t = 1$.)

We use the shorthand notation \mathbb{E}^t instead of \mathbb{E}^{μ_t} .

We shall in fact obtain the following stronger result:

Proposition 5.3. *Let $\mu \in \mathcal{P}(B)$ satisfy the locally uniform first-moment deviation inequality and assume μ has a finite first moment, then there exists a neighborhood of μ in $\mathcal{P}(B)$, say \mathcal{N} , and a constant C such that for all μ_0 and μ_1 in \mathcal{N} and for all $n \geq 1$ then*

$$\frac{1}{n} \mathbb{E}^1[d(id, Z_n)] - \frac{1}{n} \mathbb{E}^0[d(id, Z_n)] \leq C \nu(\mu_0, \mu_1). \quad (5.23)$$

The proof yields an explicit value for the constant C in Proposition 5.3, namely

$$C = 2(1 + \sup_{t \in [0, 1]} \nu(\mu_t, \mu)) \chi_1(\mu; d) + 4 \sup_{t \in [0, 1]} \tau_1(\mu_t). \quad (5.24)$$

We start with a simple observation:

Lemma 5.4. *Let $\mu \in \mathcal{P}(B)$ and $\mu_0 \in \mathcal{P}(B)$ and assume μ has a finite first moment. Then*

$$\chi_1(\mu_0; d) \leq (1 + \nu(\mu_0, \mu)) \chi_1(\mu; d).$$

Proof. By definition of $\nu(\mu_0, \mu)$, we have $\mu_0(a) \leq (1 + \nu(\mu_0, \mu)) \mu(a)$ for all a . The inequality in the Lemma follows. \square

The next steps of the proof rely on the Girsanov formula. Let $\mu_0 \in \mathcal{P}(B)$. Then the restriction of \mathbb{P}^{μ_0} to the σ -field \mathcal{F}_n is absolutely continuous with respect to the restriction of \mathbb{P}^μ with Radon-Nikodym derivative equal to $\Pi_{j=1}^n \frac{\mu_0(X_j)}{\mu(X_j)}$. Therefore the following **Girsanov formula** holds for any non-negative measurable function $F : G^n \rightarrow \mathbb{R}_+$:

$$\mathbb{E}^0[F(X_1, \dots, X_n)] = \mathbb{E}^\mu[F(X_1, \dots, X_n) \Pi_{j=1}^n \frac{\mu_0(X_j)}{\mu(X_j)}]. \quad (5.25)$$

Applying (5.25) to μ_t , we get that for any non-negative measurable function $F : G^n \rightarrow \mathbb{R}_+$:

$$\mathbb{E}^t[F(X_1, \dots, X_n)] = \mathbb{E}^0[F(X_1, \dots, X_n) \Pi_{j=1}^n \frac{\mu_t(X_j)}{\mu_0(X_j)}]. \quad (5.26)$$

In particular

$$\mathbb{E}^t[d(id, Z_n)] = \mathbb{E}^0[d(id, Z_n) \Pi_{j=1}^n \frac{\mu_t(X_j)}{\mu_0(X_j)}]. \quad (5.27)$$

Let us take the derivative in t in equation (5.27). This is justified since the expectation w.r.t. \mathbb{E}^0 in (5.27) is in fact a polynomial in t . We get that

$$\begin{aligned} \frac{d}{dt} \mathbb{E}^t[d(id, Z_n)] &= \sum_{k=1}^n \mathbb{E}^0[d(id, Z_n) \frac{\mu_1(X_k) - \mu_0(X_k)}{\mu_t(X_k)} \Pi_{j=1}^n \frac{\mu_t(X_j)}{\mu_0(X_j)}] \\ &= \sum_{k=1}^n \mathbb{E}^0[d(id, Z_n) \nu_t(X_k) \Pi_{j=1}^n \frac{\mu_t(X_j)}{\mu_0(X_j)}]. \end{aligned}$$

Using the Girsanov formula again (but in the other direction!), we deduce that

$$\frac{d}{dt} \mathbb{E}^t[d(id, Z_n)] = \sum_{k=1}^n \mathbb{E}^t[d(id, Z_n) \nu_t(X_k)]. \quad (5.28)$$

Thus Proposition 5.3 will come as a consequence of the following

Lemma 5.5. *Let $\mu \in \mathcal{P}(B)$ and let $f : B \rightarrow \mathbb{R}$ be bounded and such that $\sum_{a \in B} f(a) \mu(a) = 0$. Then for all $n \geq 1$ and $1 \leq k \leq n$ we have*

$$|\mathbb{E}^\mu[d(id, Z_n) f(X_k)]| \leq (\max_{a \in B} |f(a)|) (2\mathbb{E}^\mu[d(id, X_1)] + 4\mathbb{E}^\mu[(id, Z_n)_{Z_{k-1}}]). \quad (5.29)$$

Proof of Lemma 5.5. By assumption $f(X_k)$ is centered under \mathbb{P}^μ .

We use the same replacement trick as in Section 4.1. Let X'_k be a random variable with distribution μ and independent of (X_1, \dots, X_n) . Let $Z_n^{(k)} := Z_{k-1} X'_k (Z_k^{-1} Z_n)$ be the element of G we obtain when replacing the k -th increment of the random walk by X'_k . Then $f(X_k)$ is independent of $Z_n^{(k)}$. Therefore $\mathbb{E}^\mu[d(id, Z_n^{(k)}) f(X_k)] = 0$ and

$$\mathbb{E}^\mu[d(id, Z_n) f(X_k)] = \mathbb{E}^\mu[(d(id, Z_n) - d(id, Z_n^{(k)})) f(X_k)],$$

and

$$|\mathbb{E}^\mu[d(id, Z_n) f(X_k)]| \leq (\max_{a \in B} |f(a)|) \mathbb{E}^\mu[|d(id, Z_n) - d(id, Z_n^{(k)})|]. \quad (5.30)$$

We next bound $\mathbb{E}^\mu[|d(id, Z_n) - d(id, Z_n^{(k)})|]$ in terms of Gromov products.

Choose x, x', y and z in G and observe that

$$\begin{aligned} &d(id, yxz) - d(id, yx'z) \\ &= d(id, yxz) - d(id, y) - d(yx'z, y) + 2(id, yx'z)_y \\ &\leq d(id, xz) - d(id, x'z) + 2(id, yx'z)_y. \end{aligned}$$

But $d(id, xz) - d(id, x'z) \leq d(x', x)$ and therefore

$$d(id, yxz) - d(id, yx'z) \leq d(x', x) + 2(id, yx'z)_y.$$

Applying this last inequality with $y = Z_{k-1}$, $x = X_k$, $x' = X'_k$ and $z = Z_k^{-1} Z_n$, we get that

$$|d(id, Z_n) - d(id, Z_n^{(k)})| \leq d(X'_k, X_k) + 2(id, Z_n)_{Z_{k-1}} + 2(id, Z_n^{(k)})_{Z_{k-1}}.$$

Observe that $d(X'_k, X_k)$ is bounded by $d(id, X_k) + d(id, X'_k)$. Therefore

$$|d(id, Z_n) - d(id, Z_n^{(k)})| \leq d(id, X_k) + d(id, X'_k) + 2(id, Z_n)_{Z_{k-1}} + 2(id, Z_n^{(k)})_{Z_{k-1}}. \quad (5.31)$$

Observe that both X_k and X'_k have the same law as X_1 . Besides $(id, Z_n^{(k)})_{Z_{k-1}}$ and $(id, Z_n)_{Z_{k-1}}$ have the same law. Taking expectations in (5.31) yields

$$\mathbb{E}^\mu[|d(id, Z_n) - d(id, Z_n^{(k)})|] \leq 2\mathbb{E}^\mu[d(id, X_1)] + 4\mathbb{E}^\mu[(id, Z_n)_{Z_{k-1}}],$$

which concludes the proof of the Lemma. \square

End of the proof of Proposition 5.3. Choose \mathcal{N} such that $\sup_{\mu_0, \mu_1 \in \mathcal{N}} \sup_{t \in [0,1]} \nu(\mu_t, \mu) + \tau_1(\mu_t) < \infty$. Apply Lemma 5.5 to μ_t and ν_t and Lemma 5.4 (with μ_0 replaced by μ_t) to get that

$$\mathbb{E}^t[d(id, Z_n)\nu_t(X_k)] \leq \nu(\mu_0, \mu_1)(2(1 + \nu(\mu_t, \mu))\mathbb{E}^\mu[d(id, X_1)] + 4\tau_1(\mu_t)),$$

and deduce from formula (5.28) that

$$\frac{1}{n} \frac{d}{dt} \mathbb{E}^t[|Z_n|] \leq C \nu(\mu_0, \mu_1),$$

with C given by (5.24). The Proposition follows at once. \square

5.5 Proof of Theorem 5.2.

In the first part of the proof, we argue that Theorem 4.1 implies that the limit defining $\sigma(\nu, \mu; d)$ in (5.22) exists.

The rest of the proof of Theorem 5.2 follows the same strategy as in [Mat14]. As in (5.28), one may use the Girsanov formula to write the derivative of $\mathbb{E}^t[d(id, Z_n)]$ as a covariance, divide by n and let n tend to ∞ . This leads to a correct guess for the identification of the derivative but it remains to explain how to exchange the two limits as n tend to ∞ and λ tends to 0. The justification we find in [Mat14] uses Gaussian integration by parts and the full strength of the C.L.T. (not just the existence of the variance).

For $n \geq 1$, let us define

$$M_n := \sum_{j=1}^n \nu(X_j); M_0 = 0.$$

Recall that $\nu(a)\mu(a)$ is the derivative at $t = 0$ of the function $\mu_t(a)$. Since all measures μ_t are probability measures, we must have $\sum_a \nu(a)\mu(a) = 0$. Therefore the sequence of random variables $(M_n)_{n \in \mathbb{N}}$ is a centered martingale under \mathbb{E}^μ . In particular it satisfies the C.L.T.

Let us now check that the limit defining $\sigma(\nu, \mu; d)$ in (5.22) exists. First observe that, since M_n is centered, then $\mathbb{E}^\mu[d(id, Z_n)M_n] = \mathbb{E}^\mu[(d(id, Z_n) - \mathbb{E}^\mu[d(id, Z_n)])M_n]$ is the covariance of $d(id, Z_n)$ and M_n .

Let $a \in \mathbb{R}$. We define a D.A.C. $\mathcal{Q}^a := (Q_n^a := d(id, Z_n) + aM_n)_{n \in \mathbb{N}^*}$. Note that since ν is bounded and since we are assuming that μ has a finite second-moment, then \mathcal{Q}^a has a finite second moment. The defect of \mathcal{Q}^a coincides with the defect of the length D.A.C. and equals

$$\Psi_{n,m}^a = -2(id, Z_{n+m})_{Z_n};$$

see the discussion after Definition 3.1. Since we are assuming the second-moment deviation inequality for μ , then \mathcal{Q}^a also satisfies the second-moment deviation inequality in the sense of Definition 3.1.

A first application of Theorem 4.1 to the D.A.C. \mathcal{Q}^0 yields the existence of the limit $\lim_n \frac{1}{n} \mathbb{V}^\mu[d(id, Z_n)]$. It is clear that the limit $\lim_n \frac{1}{n} \mathbb{V}^\mu[M_n]$ also exists. Applying now Theorem 4.1 to $\mathcal{Q}^{1/2}$, we deduce the existence of the limit $\lim_n \frac{1}{n} \mathbb{E}^\mu[d(id, Z_n)M_n] = \lim_n \frac{1}{n} (\mathbb{V}^\mu[Q_n^{1/2}] - \frac{1}{4} \mathbb{V}^\mu[M_n^2] - \mathbb{V}^\mu[d(id, Z_n)])$.

The rest of the proof of Theorem 5.2 follows the same strategy as in [Mat14].

We are assuming that there exists a neighborhood of μ in $\mathcal{P}(B)$, say \mathcal{N} , on which all measures satisfy the first-moment deviation inequality with the same constant $\tau_1(\mu)$. Under the assumption $\sup_{t \in [0,1]} \sup_{a \in B} |\frac{1}{t} \log \frac{\mu_t(a)}{\mu_0(a)} - \nu(a)| < \infty$, we see that $\nu(\mu_t, \mu_0)$ tends to 0 as t tends to 0. Therefore, for all sufficiently small t , we have $\mu_t \in \mathcal{N}$.

Lemma 3.4 applied to the D.A.C. \mathcal{Q}^0 and the assumption of locally uniform first-moment inequality imply Lemma 3.1 in [Mat14].

The Central Limit Theorem 4.2 applied to the family of D.A.C. \mathcal{Q}^a implies the joint Central Limit Theorem for the vector $\frac{1}{\sqrt{n}}(d(id, Z_n) - n\ell(\mu; d), M_n)$ as in Proposition 3.2 (i) of [Mat14]; see Lemma 5.6 below.

Note that the variance upper bound needed to apply Theorem 2.3 (assumption (ii)) follows from Theorem 4.3 here.

Also note that Theorem 2.3 was written for a measure μ with finite support. The details to adapt it to unbounded supports are straightforward.

Lemma 5.6. *Let $(A_n), (B_n)$ be sequences of random variables. Then the random vectors (A_n, B_n) converge in distribution to the random vector (A, B) if and only if (B_n) converges in distribution to B and for each $a \in \mathbb{R}$ the random variables $A_n + aB_n$ converge in distribution to $A + aB$.*

Proof. By Lévy's Theorem, (A_n, B_n) converge in distribution to (A, B) if and only if for every $a, b \in \mathbb{R}$ we have $\mathbb{E}^\mu[e^{i(bA_n + aB_n)}] \rightarrow \mathbb{E}^\mu[e^{i(bA + aB)}]$. By (the other direction of) Lévy's Theorem this happens if and only if, for every $a, b \in \mathbb{R}$, $bA_n + aB_n$ converges in distribution to $bA + aB$, and the conclusion easily follows. \square

6 Lipschitz regularity of the entropy

One may deduce the Lipschitz continuity of the entropy from Theorem 5.1 using the identification of the entropy as a rate of escape in the so-called Green metric, see paragraph 6.1 below. This argument is reminiscent of the proof in part 4 of [Mat14]. Because the Green metric is a true distance (i.e. symmetric) only when μ is itself symmetric, we have to restrict ourselves to symmetric measures.

In the sequel, $\mathcal{P}_s(B)$ will denote the set of symmetric probability measures with support B .

We shall assume that G is finitely generated and further impose that G is non-amenable.

Theorem 6.1. *Assume that G is finitely generated and non-amenable. Let B be a (finite or infinite) symmetric generating subset of G and choose a symmetric measure $\mu \in \mathcal{P}_s(B)$. Assume that there exists a neighborhood of μ in $\mathcal{P}_s(B)$, say \mathcal{N}_0 , such that the first-moment deviation inequality (5.20) holds uniformly for $\mu \in \mathcal{N}_0$ and also uniformly with respect to all the Green metrics $d_G^{\mu'}$ associated with a measure μ' in \mathcal{N}_0 . Assume that μ has a finite first moment.*

Then there exists a neighborhood of μ in $\mathcal{P}_s(B)$, say \mathcal{N} , such that the function $\mu \rightarrow h(\mu)$ is Lipschitz continuous on \mathcal{N} .

Theorem 6.2. *Assume that G is finitely generated and non-amenable. Let B be a (finite or infinite) symmetric generating subset of G and choose a symmetric measure $\mu \in \mathcal{P}_s(B)$. Assume that μ has a finite second moment, satisfies the second-moment deviation inequality and the locally uniform first-moment deviation inequality in the Green metric d_G^μ .*

Then the function $\mu_0 \rightarrow h(\mu_0)$ is differentiable at $\mu_0 = \mu$ in the following sense: Let $(\mu_t, t \in [0, 1])$ be a curve in $\mathcal{P}(B)$ such that $\mu_0 = \mu$ and, for all $a \in B$, the function $t \rightarrow \log \mu_t(a)$ has a derivative at $t = 0$, say $\nu(a)$. We assume that ν is bounded on B and also that $\sup_{t \in [0,1]} \sup_{a \in B} |\frac{1}{t} \log \frac{\mu_t(a)}{\mu_0(a)} - \nu(a)| < \infty$.

∞ . Then the limit of $\frac{1}{t}(h\mu_t) - h(\mu)$ as t tends to 0 exists. Besides this limit coincides with the covariance

$$\sigma_{\mathcal{G}}(\nu, \mu) := \lim_n \frac{1}{n} \mathbb{E}^\mu [d_{\mathcal{G}}^\mu(id, Z_n) \left(\sum_{j=1}^n \nu(X_j) \right)]. \quad (6.32)$$

6.1 Green metrics

Let us first recall some useful facts about the Green metric.

Let G be a non-amenable group. Let μ be a symmetric probability measure on G whose support generates the whole group.

We recall that there exists a constant, $\rho_\mu < 1$ - the **spectral radius** - such that

$$\mu^n(z) \leq (\rho_\mu)^n, \quad (6.33)$$

for all $n \geq 0$ and $z \in G$, see [Woe00].

The Green function is defined by

$$G^\mu(x) := \sum_{n=0}^{\infty} \mu^n(x).$$

Because of (6.33), the series defining G^μ does converge. The Green distance between points x and y in G is then

$$d_{\mathcal{G}}^\mu(x, y) := \log G^\mu(id) - \log G^\mu(x^{-1}y).$$

It follows from (6.33), that $d_{\mathcal{G}}^\mu$ is equivalent to word metrics on G .

We may equivalently express $d_{\mathcal{G}}^\mu$ in terms of the hitting probabilities of the random walk: for a given trajectory $\omega \in \Omega$ and $z \in \Gamma$, let

$$T_z(\omega) = \inf\{n \geq 0; Z_n(\omega) = z\}$$

be the hitting time of z by ω . Observe that $T_z(\omega)$ may be infinite.

Define $F^\mu(z) := \mathbb{P}^\mu[T_z < \infty]$. Then

$$d_{\mathcal{G}}^\mu(id, z) = -\log F^\mu(z),$$

as can be easily checked using the Markov property.

It is not difficult to show that this indeed defines a proper left-invariant distance on G , see [BB07] and [BHM11] for the details. Observe that $d_{\mathcal{G}}^\mu$ need not be geodesic.

In [BHM08] (see also [BP94]), we proved that

$$h(\mu) = \ell(\mu; d_{\mathcal{G}}^\mu). \quad (6.34)$$

It makes sense to define the Green metric through the Green function as soon as the random walk is transient. The identification (6.34) is also valid in this extended framework but we shall not need it here.

6.2 Fluctuations of the Green metric

Our first aim is to control the fluctuations between two Green metric, say $d_G^{\mu_0}$ and $d_G^{\mu_1}$.

We use the same notation as in the beginning of Part 5.4: Let μ_0 and μ_1 belong to $\mathcal{P}_s(B)$. For $t \in [0, 1]$ and $a \in B$, we define $\mu_t(a) := \mu_0(a) + t(\mu_1(a) - \mu_0(a))$ and $\nu_t(a) = (\mu_1(a) - \mu_0(a))/\mu_t(a)$. Then $\nu(\mu_0, \mu_1) = \sup_{a \in B} \sup_{t \in [0, 1]} |\nu_t(a)|$.

Proposition 6.3. *Let G be a finitely generated non-amenable group equipped with a word metric denoted with d . Let B be a symmetric generating sub-set of G .*

For any μ in $\mathcal{P}_s(B)$, there exists $\varepsilon_\mu > 0$ and k_μ such that for any two symmetric measures μ_0 and μ_1 in $\mathcal{P}_s(B)$ satisfying

$$\nu(\mu, \mu_0) + \nu(\mu, \mu_1) \leq \varepsilon_\mu, \quad (6.35)$$

then

$$|d_G^{\mu_1}(id, z) - d_G^{\mu_0}(id, z)| \leq k_\mu \nu(\mu_0, \mu_1) d(id, z), \quad (6.36)$$

for all $z \in G$.

We use the shorthand notation \mathbb{E}^t (resp. \mathbb{P}^t) instead of \mathbb{E}^{μ_t} (resp. \mathbb{P}^{μ_t}) and d_G^t instead of $d_G^{\mu_t}$ and F^t instead of F^{μ_t} .

The proof of Proposition 6.3 is based on the following Lemma

Lemma 6.4. *In the context of Proposition 6.3 and with the same notation, then the conditional expectation of T_z given it is finite satisfies*

$$\mathbb{E}^t[T_z | T_z < \infty] \leq k_\mu d(id, z), \quad (6.37)$$

for all $t \in [0, 1]$ and $z \in G$ for some constant k_μ .

Proof. Let $\mu \in \mathcal{P}_s(B)$. Recall that $\rho_\mu < 1$.

Let $\rho' := \frac{1}{2}(1 + \rho)$. Choose ε_μ so small that measures satisfying (6.35) are such that $\rho_{\mu_t} \leq \rho'$ for all $t \in [0, 1]$.

Also assume that ε_μ is such that there exists $\gamma > 0$ such that, for all $t \in [0, 1]$ and for all $z \in G$ then

$$\mathbb{P}^t[T_z < \infty] \geq \gamma^{d(id, z)}.$$

Both these conditions are ensured by the following: since B generates G and since G is finitely generated, then there exists a finite sub-set of B , say \tilde{B} , that generates G . The uniform upper bound on the spectral radius as well as the uniform lower bound on the probability of hitting a point z in G are both obtained once we choose ε_μ such that all measures μ_t are uniformly bounded from below on \tilde{B} .

By (6.33), we have

$$\mathbb{P}^t[T_z = k] \leq \mathbb{P}^t[Z_k = z] \leq (\rho')^k.$$

Therefore, for any $c > 0$,

$$\mathbb{E}^t[T_z | T_z < \infty] \leq c d(id, z) + \gamma^{-d(id, z)} \sum_{k \geq c d(id, z)} k (\rho')^k.$$

It only remains to choose c large enough so that $\gamma^{-d(id, z)} \sum_{k \geq c d(id, z)} k (\rho')^k \leq 1$. \square

Proof of Proposition 6.3. Let N be an integer.
The Girsanov formula (5.26) implies that

$$\mathbb{P}^t[T_z \leq N] = \mathbb{E}^0[\mathbf{1}_{T_z \leq N} \Pi_{j=1}^N \frac{\mu_t(X_j)}{\mu_0(X_j)}].$$

Taking the derivative with respect to t , we get that

$$\frac{d}{dt} \mathbb{P}^t[T_z \leq N] = \mathbb{E}^t[\mathbf{1}_{T_z \leq N} \sum_{j=1}^N \nu_t(X_j)].$$

The martingale property implies that

$$\mathbb{E}^t[\mathbf{1}_{T_z \leq N} \sum_{j=1}^N \nu_t(X_j)] = \mathbb{E}^t[\mathbf{1}_{T_z \leq N} \sum_{j=1}^{T_z} \nu_t(X_j)],$$

so that

$$\frac{d}{dt} \mathbb{P}^t[T_z \leq N] = \mathbb{E}^t[\mathbf{1}_{T_z \leq N} \sum_{j=1}^{T_z} \nu_t(X_j)],$$

and

$$\frac{d}{dt} \mathbb{P}^t[T_z \leq N] \leq \nu(\mu_0, \mu_1) \mathbb{E}^t[T_z \mathbf{1}_{T_z \leq N}].$$

Choose N large enough so that $\mathbb{P}^t[T_z \leq N] \neq 0$ for all t , and use Lemma 6.4 to get that

$$\frac{1}{\mathbb{P}^t[T_z \leq N]} \frac{d}{dt} \mathbb{P}^t[T_z \leq N] \leq \nu(\mu_0, \mu_1) k_\mu d(id, z) \frac{\mathbb{P}^t[T_z < \infty]}{\mathbb{P}^t[T_z \leq N]},$$

and therefore

$$\log \mathbb{P}^1[T_z \leq N] - \log \mathbb{P}^0[T_z \leq N] \leq \nu(\mu_0, \mu_1) k_\mu d(id, z) \int_0^1 \frac{\mathbb{P}^t[T_z < \infty]}{\mathbb{P}^t[T_z \leq N]} dt.$$

We now let N tend to $+\infty$. Observe that there exist N_0 and ε such that, for all t , then $\mathbb{P}^t[T_z \leq N] \geq \varepsilon$ for all $N \geq N_0$. Thus we may apply the dominated convergence Lemma to deduce that

$$\log \mathbb{P}^1[T_z < \infty] - \log \mathbb{P}^0[T_z < \infty] \leq \nu(\mu_0, \mu_1) k_\mu d(id, z).$$

Exchanging the roles of μ_0 and μ_1 leads to

$$|\log \mathbb{P}^1[T_z < \infty] - \log \mathbb{P}^0[T_z < \infty]| \leq \nu(\mu_0, \mu_1) k_\mu d(id, z).$$

□

Proof of Theorem 6.1. Write that

$$\begin{aligned} h(\mu_1) - h(\mu_0) &= \ell(\mu_1; d_{\mathcal{G}}^1) - \ell(\mu_0; d_{\mathcal{G}}^0) \\ &= (\ell(\mu_1; d_{\mathcal{G}}^1) - \ell(\mu_0; d_{\mathcal{G}}^1)) + (\ell(\mu_0; d_{\mathcal{G}}^1) - \ell(\mu_0; d_{\mathcal{G}}^0)) := I + II. \end{aligned}$$

We argue that both terms I and II are bounded by $C\nu(\mu_0, \mu_1)$ for some C , uniformly in a small enough neighborhood of μ in $\mathcal{P}_s(B)$.

The first term I is handled as in Part 5.4: we use the assumption that the first-moment deviation inequality is uniform in a neighborhood of μ w.r.t. both the driving measure of the random walk and the one defining the Green metric.

For the second term II, we rely on Proposition 6.3. We have

$$|d_{\mathcal{G}}^1(id, Z_n) - d_{\mathcal{G}}^0(id, Z_n)| \leq k_{\mu} \nu(\mu_0, \mu_1) d(id, Z_n).$$

Taking the expectation with respect to \mathbb{E}^0 , dividing by n and letting n tend to ∞ , we get that

$$|\ell(\mu_0; d_{\mathcal{G}}^1) - \ell(\mu_0; d_{\mathcal{G}}^0)| \leq k_{\mu} \nu(\mu_0, \mu_1) \ell(\mu_0; d).$$

It then suffices to note that $\ell(\mu_0; d) \leq \mathbb{E}^0[d(id, X_1)]$ is bounded on a neighborhood of μ , see Lemma 5.4. \square

Proof of Theorem 6.2.

Recall that $h(\mu) = \ell(\mu; d_{\mathcal{G}}^{\mu})$ and write that

$$\frac{1}{t}(h(\mu_t) - h(\mu_0)) = \frac{1}{t}(h(\mu_t) - \ell(\mu_t; d_{\mathcal{G}}^{\mu})) + \frac{1}{t}(\ell(\mu_t; d_{\mathcal{G}}^{\mu}) - \ell(\mu; d_{\mathcal{G}}^{\mu})).$$

We apply Theorem 5.2 in the Green metric $d_{\mathcal{G}}^{\mu}$ and deduce that the second term converges to $\sigma_{\mathcal{G}}(\nu, \mu)$. The first term goes to 0 by Proposition 4.1 in [Mat14]. \square

Part II

Getting deviation inequalities

7 Acylindrically hyperbolic groups

Recall that a geodesic metric space is (Gromov-)hyperbolic if there exists $\delta \geq 0$ so that for any geodesic triangle $[p, q] \cup [q, r] \cup [r, p]$ and each $x \in [p, q]$ there exists $y \in [q, r] \cup [r, p]$ so that $d_X(x, y) \leq \delta$. A finitely generated group G is hyperbolic if some (equivalently, any) Cayley graph of G is hyperbolic.

The groups for which we can prove deviation inequalities are the so-called *acylindrically hyperbolic groups*. Such class of groups vastly generalises the class of hyperbolic groups and includes non-elementary relatively hyperbolic groups, Mapping Class Groups, $Out(F_n)$, many groups acting on CAT(0) cube complexes (e.g. right-angled Artin groups that do not split as a direct product), possibly infinitely presented small cancellation groups, and many subgroups of the above. In particular, we will refine and generalise the results in [Sis14b], some of whose techniques are used here as well.

Acylindrical hyperbolicity is defined in terms of an "interesting enough" action on some hyperbolic space. "Interesting enough" means in this case acylindrical and non-elementary, see Section 8 for the definitions. Roughly speaking, acylindricity says that the coarse stabiliser of any two far away points is finite, and being non-elementary is a non-triviality-type condition. Acylindrically hyperbolic groups have been defined by Osin who showed in [Osi14] that several approaches to groups that exhibit rank one behaviour [BF02, Ham08, DGO11, Sis11] are all equivalent. Acylindrical hyperbolicity has strong consequences: Every acylindrically hyperbolic group is SQ-universal (in particular it has uncountably many pairwise non-isomorphic quotients), it contains free normal subgroups [DGO11], it contains Morse elements and hence all its asymptotic cones have cut-points [Sis14a], and its bounded cohomology is infinite dimensional in degrees 2 [HO13] and 3 [FPS13]. Moreover, if an acylindrically hyperbolic group does not contain finite normal subgroups, then its reduced C^* -algebra is simple [DGO11] and every commensurating endomorphism is an inner automorphism [AMS13].

We will in fact not only consider word metrics on acylindrically hyperbolic groups but more generally metrics coming from actions with certain properties, see Definition 10.1. In particular, this covers the metric that an acylindrically hyperbolic group inherits from the hyperbolic space it acts on. Perhaps even more interestingly, it covers the metric coming from the action of a Mapping Class Group on the corresponding Teichmüller space endowed with either the Teichmüller or the Weil-Petersson metric, and the metric inherited by the fundamental group of a finite-volume hyperbolic n -manifold (an example of relatively hyperbolic group) on \mathbb{H}^n .

8 Preliminaries

A discrete path is an ordered sequence of points $\alpha = (w_i)_{k_1 \leq i \leq k_2}$ in a metric space Y . Its length $l_Y(\alpha)$ is defined as $\sum d(w_i, w_{i+1})$. The notions of Lipschitz and quasi-geodesic discrete paths are defined regarding a discrete path as a map from an interval in \mathbb{Z} to Y . We will often omit the adjective discrete.

8.1 Acylindrical actions

Let G act by isometries on the metric space X . The action is called **acylindrical** if for every $r \geq 0$ there exist $R, N \geq 0$ so that whenever $x, y \in X$ satisfy $d_X(x, y) \geq R$ there are at most N elements $g \in G$ so that $d_X(x, gx), d_X(y, gy) \leq r$. Also, we will say that the action is **non-elementary** if orbits are unbounded and G is not virtually cyclic (cfr. [Osi14, Theorem 1.1]).

When an action of a group G on the metric space X and a word metric d_G on G have been fixed, we denote by diam^* the diameter, by $B^*(\cdot, R)$ a ball of radius R and by N_t^* a neighborhood of radius t , where $*$ can be either G or X depending on which metric we are using to define the given notion.

We will need the following lemma about acylindrical actions (a similar lemma is exploited in [Sis14a]).

Lemma 8.1. *Let the finitely generated group G act acylindrically on the hyperbolic geodesic metric space X . Let $\pi : G \rightarrow X$ be an orbit map with basepoint x_0 , and endow G with a word metric d_G .*

Then there exists L and a non-decreasing function f so that for each $l_1, l_2 \geq 0$, each $t \geq 0$ and whenever $x, y \in Gx_0$ satisfy $d_X(x, y) \geq L + l_1 + l_2$, we have

$$\text{diam}^G(\pi^{-1}(B^X(x, l_1)) \cap N_t^G(\pi^{-1}(B^X(y, l_2))) \leq f(t).$$

Proof. First of all, we choose the constants. Let δ be the hyperbolicity constant of X . Let R, N be as in the definition of acylindrical action with $r = 4\delta + 1$. Finally, let $L = R + 6\delta + 1$.

Up to applying an element of G , we can and will assume $x = x_0$ throughout the proof. Fix some t from now on. Let $\{h_i\}_{i=1, \dots, k}$ be the finitely many elements of $B^G(id, t)$ and for each i let γ_i be a geodesic in X from x_0 to $h_i x_0$.

Claim 1: There exist only finitely many $g \in G$ so that $\text{diam}^X(N_{2\delta}^X(\gamma_i) \cap g\gamma_j) \geq R$ for some i, j . In particular, the diameter of the set of such g 's is bounded by, say, $f(t)$.

Proof of Claim 1. Since there are only finitely many h_i 's, we can fix i, j . Suppose by contradiction that there exist infinitely many distinct elements $g_k, k = 0, 1, \dots$, so that $\text{diam}^X(N_{2\delta}^X(\gamma_i) \cap g_k\gamma_j) \geq R$. Hence, for every k there exist points $p_k, q_k \in \gamma_j$ and $p'_k, q'_k \in \gamma_i$ so that $d_X(g_k p_k, p'_k), d_X(g_k q_k, q'_k) \leq 2\delta$ and $d_X(p_k, q_k) \geq R$. By compactness of γ_i and γ_j , we can pass to a subsequence of $\{g_k\}$ and assume that for every k_1, k_2 we have that each of $d_X(p_{k_1}, p_{k_2}), d_X(q_{k_1}, q_{k_2}), d_X(p'_{k_1}, p'_{k_2}), d_X(q'_{k_1}, q'_{k_2})$ is at most $1/2$. In particular, for every k we have $d_X(p_0, g_0^{-1} g_k p_0) = d_X(g_0 p_0, g_k p_0) \leq 4\delta + 1$ and similarly $d_X(q_0, g_0^{-1} g_k q_0) \leq 4\delta + 1$. But this is a contradiction because acylindricity implies that there can be

only finitely many a so that $d_X(p_0, ap_0), d_X(q_0, aq_0) \leq 4\delta + 1$ (since $d_X(p_0, q_0) \geq R$). This completes the proof of the claim. \square

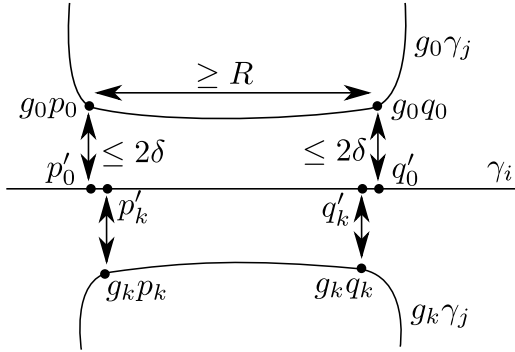


Figure 1: Claim 1. Translates of γ_j stay close to the same subgeodesic of γ_i .

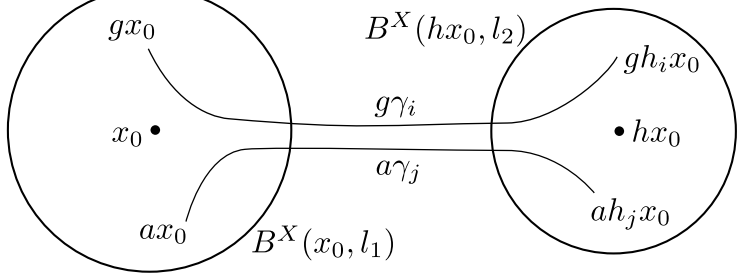


Figure 2: Claim 2. $g\gamma_i$ and $h\gamma_j$ fellow-travel between $B^X(x_0, l_1)$ and $B^X(hx_0, l_2)$.

For $h \in G$ and $l_1, l_2 \geq 0$, denote $A_{l_1, l_2}(h) = \pi^{-1}(B^X(x_0, l_1)) \cap N_t^G(\pi^{-1}(B^X(hx_0, l_2)))$. Let $h \in G$ be so that $d_X(x_0, hx_0) \geq L + l_1 + l_2$. Assume that $A_{l_1, l_2}(h)$ is non-empty, for otherwise there is nothing to prove, and let $g \in A_{l_1, l_2}(h)$. Notice that there exists i so that $gh_i x_0 \in B^X(hx_0, l_2)$. Similarly, for each $a \in A_{l_1, l_2}(h)$ there exists $j = j(a)$ so that $ah_j x_0 \in B^X(hx_0, l_2)$.

Claim 2: For any $a \in A_{l_1, l_2}(h)$ we have $\text{diam}^X(N_{2\delta}^X(g\gamma_i) \cap a\gamma_j) \geq R$, where $j = j(a)$.

Proof of Claim 2. Consider a geodesic quadrangle in X containing $g\gamma_i$ and $a\gamma_j$ with vertices $gx_0, ax_0, gh_i x_0, ah_j x_0$. Notice that the first two vertices belong to $B_1 = B^X(x_0, l_1)$ while the other vertices belong to $B_2 = B^X(hx_0, l_2)$, and by our hypothesis on h the distance between B_1 and B_2 is at least L . Also, considering a triangle with vertices x_0, gx_0, ax_0 it is readily seen that the geodesic $[gx_0, ax_0]$ is contained in the δ -neighborhood of B_1 , and that, similarly, $[gh_i x_0, ah_j x_0]$ is contained in the δ -neighborhood of B_2 . Consider now a subgeodesic γ of $a\gamma_j$ of length at least $L - 6\delta - 1$ that does not intersect the 3δ -neighborhood of $B_1 \cup B_2$. By 2δ -thinness of geodesic quadrangles, any point of γ is 2δ close to $g\gamma_i$, since it cannot be 2δ -close to either $[gx_0, ax_0]$ or $[gh_i x_0, ah_j x_0]$. Hence, $\text{diam}^X(N_{2\delta}^X(g\gamma_i) \cap a\gamma_j) \geq L - 6\delta - 1 = R$, as required. \square

In view of Claim 2 we get that, for each $a \in A_{l_1, l_2}(h)$, $g^{-1}a$ belongs to the finite set given by Claim 1. In particular $\text{diam}^G(A_{l_1, l_2}(h)) \leq f(t)$, as required. \square

9 Linear progress with exponential decay

When a group G acting on a hyperbolic space X is fixed, we will implicitly make a choice of basepoint $x_0 \in X$ and, to simplify the notation, write $d_X(g, h)$ instead of $d_X(gx_0, hx_0)$ when $g, h \in G$.

We say that a group is *non-elementary* if it does not contain a cyclic subgroup of finite index. When μ is a measure on the group G , the subgroup generated by the support of μ is the smallest subgroup of G containing the support of μ .

Theorem 9.1. *Let G act acylindrically on the geodesic hyperbolic space X . Then any measure μ_0 with exponential tail whose support generates a non-elementary subgroup that acts with unbounded orbits on X has a neighborhood, say \mathcal{N} , so that there exists C with the following property. For any $\mu \in \mathcal{N}$, any positive integer n and any $g_0 \in G$ we have*

$$\mathbb{P}^\mu[d_X(id, g_0 Z_n) - d_X(id, g_0) \leq n/C] \leq Ce^{-n/C}.$$

In particular, the rate of escape measured in the metric d_X of the random walk driven by μ is strictly positive.

Remark 9.2. The condition on the support of μ_0 will also appear in Theorems 10.6, 12.1. Notice that if G acts non-elementarily on X such condition is weaker than requiring that the support generates G .

First of all, we remark that it is enough to show the theorem for the measure μ_0 .

Lemma 9.3. Let G, X, x_0, μ_0 be as in Theorem 9.1 and suppose that there exists C so that for any integer n we have $\mathbb{P}^{\mu_0}[d_X(id, Z_n) \leq n/C] \leq Ce^{-n/C}$. Then there exists a neighborhood \mathcal{N} of μ_0 so that for any $\mu \in \mathcal{N}$ and any positive integer n , we have $\mathbb{P}^\mu[d_X(id, Z_n) \leq n/C] \leq Ce^{-n/2C}$.

Proof. Let μ be so that $\nu(\mu, \mu_0) \leq \epsilon$, where ϵ will be chosen later. Recall that by the Girsanov formula for any non-negative measurable function $F : G^n \rightarrow \mathbb{R}_+$, we have

$$\mathbb{E}^\mu[F(X_1, \dots, X_n)] = \mathbb{E}^{\mu_0} \left[F(X_1, \dots, X_n) \prod_{j=1}^n \frac{\mu(X_j)}{\mu_0(X_j)} \right].$$

Since $\mu(a)/\mu_0(a) \leq 1 + \epsilon$ for each $a \in G$, we have

$$\mathbb{E}^\mu[F(X_1, \dots, X_n)] \leq (1 + \epsilon)^n \mathbb{E}^{\mu_0}[F(X_1, \dots, X_n)].$$

Use this inequality with $F = \mathbb{1}_A$ where A is the event “ $d_X(id, Z_n) \leq n/C$ ” yields:

$$\mathbb{P}^\mu[d_X(id, Z_n) \leq n/C] \leq C(1 + \epsilon)^n \exp^{-n/C}.$$

It is then enough to choose ϵ small enough so that $(1 + \epsilon)^n \exp^{-n/C} \leq \exp^{-n/2C}$ for ϵ small enough. \square

Fix the notation of the theorem from now on. In view of the lemma, we can fix $\mu = \mu_0$. We will write \mathbb{P} instead of \mathbb{P}^μ . All Gromov products are taken with respect to d_X , meaning that $(g, h)_k$ denotes the Gromov product $(gx_0, hx_0)_{kx_0}$ taken in X .

Proposition 9.4. There exist $C, k > 0$ with the following properties. For every $g \in G$ we have

1. For every $h \in G$

$$\mathbb{P}[(g, gZ_k h)_{id} \leq d_X(id, g) - C] \leq 1/10.$$

2.

$$\lim_{m \rightarrow \infty} \mathbb{E}[d_X(id, Z_m)] = +\infty.$$

Proof. 1) For convenience we will assume $d_X \leq d_G$, which can be arranged by rescaling the metric on X . The notation $[g, h]$ will denote any choice of a geodesic in X (not in G !) from gx_0 to hx_0 . Let δ be a hyperbolicity constant for X . We will use the following deterministic lemma.

Lemma 9.5. There exist C_0, D with the following property. For each $g, h \in G$ and $k \geq 0$ the set $A(g, h, k)$ of elements $s \in B^G(id, k)$ so that $\text{diam}^X([id, g] \cap N_{2\delta}([gs, gsh])) \geq D$ has cardinality at most $C_0 k^2$.

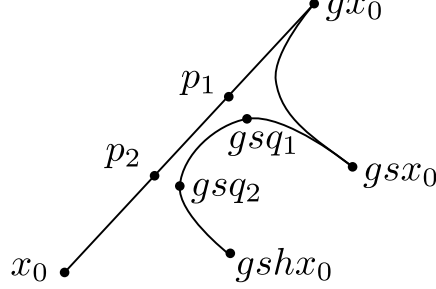
Proof. For any $D \geq 100\delta + 100$ we have that if $\text{diam}^X([id, g] \cap N_{2\delta}([gs, gsh])) \geq D$ then there exist subgeodesics $[p_1, p_2] \subseteq [id, g], [q_1, q_2] \subseteq [id, h]$ so that

1. the lengths of $[p_1, p_2], [q_1, q_2]$ are $D - 4\delta - 2$,

2. $d_X(p_i, gsq_i) \leq 4\delta + 2$,

3. $d_X(p_1, gx_0), d_X(q_1, x_0)$ are integers,
4. $d_X(p_1, gx_0), d_X(q_1, x_0) \leq k + 4\delta + 1$.

In fact, let $p'_1, p'_2 \in [id, g]$ and $gsq'_1, gsq'_2 \in [gs, gsh]$ be so that $d_X(p'_i, gsq'_i) \leq 2\delta$ and $d_X(p'_1, p'_2), d_X(q'_1, q'_2) \geq D$. If we assume that $\min\{d_X(p'_i, gx_0)\}$ is minimal, then we have $d_X(p'_i, gx_0), d_X(gsq'_i, gsx_0) \leq k + 2\delta$ by 2δ -thinness of the quadrangle with vertices $x_0, gx_0, gsx_0, gshx_0$ (as $d_X(g, gs) \leq k$).



Now, $[p'_1, p'_2]$ contains a subgeodesic $[p_1, p_2]$ with $d_X(p'_i, p_i) \in [2\delta, 2\delta + 1]$ satisfying conditions 1,3,4, and similarly for $[q_1, q_2]$. The 2δ -thinness of the quadrangle with vertices $p'_1, p'_2, gsq'_1, gsq'_2$ implies that $d_X(p_i, gsq_i) \leq 4\delta + 2$, i.e. condition 2.

What we know directly from acylindricity is that for D large enough there exists C_1 so that once we fix any subgeodesics $[p_1, p_2] \subseteq [id, g]^{\pm 1}, [q_1, q_2] \subseteq [id, h]^{\pm 1}$ of length $D - 4\delta - 2$ there are at most C_1 elements s so that $d_X(p_i, gsq_i) \leq 4\delta + 2$. For a suitable C_2 , there are at most $C_2 k^2$ choices of subgeodesics satisfying the conditions set above, so we conclude that the lemma holds for $C_0 = C_1 C_2$. \square

The deterministic lemma will be combined with the probabilistic lemma below.

Lemma 9.6. *For every L there exists k so that for every $A \subseteq G$ of cardinality at most $C_0(Lk)^2$ we have $\mathbb{P}[Z_k \in A] \leq 1/20$.*

Proof. The hypotheses on μ imply that its support generates a group containing a non-abelian free subgroup (see [Osi14, Theorem 1.1]), and in particular a non-amenable group. Hence, we have, for each $g \in G$, $\mathbb{P}[Z_k = g] \leq \rho^k$ for some $\rho < 1$ [Woe00], so the required result follows from summing over A once we choose k so that $C_0(Lk)^2 \rho^k \leq 1/20$. \square

Let us now fix some constants. Let L be so that $\mathbb{P}[d_X(id, Z_n) > Ln] \leq 1/20$ for every n , which exists because we are assuming that the measures we deal with have exponential tails. Let k be as in Lemma 9.6 for the given L . Finally let C be so that $\mathbb{P}[d_X(id, Z_k) \geq C] \leq 1/4$ and $C \geq Lk + D + 100\delta$, for δ a hyperbolicity constant for X .

We are ready to prove the required inequality, i.e. that for any $h \in G$ we have

$$\mathbb{P}[(g, gZ_k h)_{id} \leq d_X(id, g) - C] \leq 1/10. \quad (*)$$

Fix any $h \in G$. We observe that if $(g, gsh)_{id} \leq d_X(id, g) - C$, for some s with $d_X(id, s) \leq Lk$, then we have $\text{diam}(N_{2\delta}([id, g]) \cap [gs, gsh]) \geq D$.

Hence, letting $A = A(g, h, Lk)$ be as in Lemma 9.5 (in particular $\#A \leq C_0(Lk)^2$) we have

$$\mathbb{P}[(g, gZ_k h)_{id} \leq d_X(id, g) - C] \leq \mathbb{P}[d_X(id, Z_k) > Lk] + \mathbb{P}[Z_k \in A].$$

The first term is bounded by $1/20$ by the choice of L , while the second one is at most $1/20$ by Lemma 9.6, so the claim is proved.

2) Fix any $K \geq 2C$, for C as in the first part of the proposition. The argument below shows $\mathbb{E}[d_X(1, Z_n)] \geq 4(K - 2C)/5$ for each large enough n . Let us define $M = \inf\{m; d_X(1, Z_m) \geq K\}$.

The first claim is that $M < \infty$ almost surely. In fact, there exists m_0 such that $\mathbb{P}[d_X(1, Z_{m_0}) \geq 2K] = \varepsilon > 0$. Therefore, eventually one of the increments $Z_{jm_0}^{-1} Z_{(j+1)m_0}$ will exceed $2K$. If this happens for the first time for a given j , then either $d_X(1, Z_{jm_0}) \geq K$ or $d_X(1, Z_{(j+1)m_0}) \geq K$.

Let $A_{g,m}$ be the event “ $M = m$ and $Z_m = g$ ”. Then for any $n \geq m$ the events “ $A_{g,m}$ and $d_X(1, Z_n) \geq d_X(1, g) - 2C$ ” and “ $A_{g,m}$ and $d_X(1, gZ_m^{-1}Z_n) \geq d_X(1, g) - 2C$ ” coincide. Notice that $A_{g,m}$ and “ $d_X(1, gZ_m^{-1}Z_n) \geq d_X(1, g) - 2C$ ” are independent. Besides, if $n \geq m + k$, for k as in the first part of the proposition, then

$$\mathbb{P}[d_X(1, gZ_m^{-1}Z_n) \geq d_X(1, g) - 2C] \geq \frac{9}{10}$$

and

$$\mathbb{P}[A_{g,m} \text{ and } d_X(1, Z_n) \geq d_X(1, g) - 2C] \geq \frac{9}{10} \mathbb{P}[A_{g,m}],$$

which implies $\mathbb{E}[d_X(1, Z_n) | A_{g,m}] \geq 9(K - 2C)/10$ whenever g satisfies $d_X(1, g) \geq K$ (notice that $A_{g,m} = \emptyset$ if $d_X(1, g) < K$). We can now bound

$$\begin{aligned} \mathbb{E}[d_X(1, Z_n)] &\geq \sum_{m \leq n-k, d_X(1, g) \geq K} \mathbb{E}[d_X(1, Z_n) | A_{g,m}] \mathbb{P}[A_{g,m}] \\ &\geq \frac{9}{10} (K - 2C) \sum_{m \leq n-k, d_X(1, g) \geq K} \mathbb{P}[A_{g,m}] \\ &= \frac{9}{10} (K - 2C) \mathbb{P}[M \leq n - k]. \end{aligned}$$

But, since $M < \infty$ almost surely, we have $\mathbb{P}[M \leq n - k] \rightarrow 1$ as n tends to ∞ , and the proof is complete. \square

Proof of Theorem 9.1 (for $\mu = \mu_0$). Throughout the proof we denote by $A = A(g, m)$ the event “ $d_X(id, gZ_m) - d_X(id, g) \geq d_X(id, Z_m) - 2C$ ”. As noted above, this is the same event as “ $(g, gZ_m)_{id} \geq d_X(id, g) - C$ ”.

Let us start with the following claim

Claim: There exist $\lambda, \epsilon > 0$ and m so that for each $g \in G$ we have

$$\mathbb{E} \left[e^{-\lambda(d_X(id, gZ_m) - d_X(id, g))} \right] \leq 1 - \epsilon.$$

Proof of Claim. On A we have

$$d_X(id, gZ_m) - d_X(id, g) \geq d_X(id, Z_k^{-1}Z_m) - d_X(id, Z_k) - 2C,$$

while on the complement A^c we have

$$d_X(id, gZ_m) - d_X(id, g) \geq -d_X(id, Z_m) \geq -d_X(id, Z_k^{-1}Z_m) - d_X(id, Z_k).$$

So, for any $h \in G$, m and $\lambda > 0$, we have

$$\begin{aligned} &\mathbb{E} \left[e^{-\lambda(d_X(id, gZ_m) - d_X(id, g))} | Z_k^{-1}Z_m = h \right] \\ &\leq \mathbb{E} \left[e^{-\lambda 2C} e^{\lambda d_X(id, Z_k)} e^{-\lambda d_X(id, h)} \mathbb{1}_A | Z_k^{-1}Z_m = h \right] + \mathbb{E} \left[e^{\lambda d_X(id, Z_k)} e^{\lambda d_X(id, h)} \mathbb{1}_{A^c} | Z_k^{-1}Z_m = h \right] \end{aligned}$$

$$\begin{aligned}
&\leq e^{2C\lambda} \left(e^{-\lambda d_X(id, h)} \mathbb{E} \left[e^{\lambda d_X(id, Z_k)} | Z_k^{-1} Z_m = h \right] + e^{\lambda d_X(id, h)} \mathbb{E} \left[e^{\lambda d_X(id, Z_k)} \mathbb{1}_{A^c} | Z_k^{-1} Z_m = h \right] \right. \\
&\quad \left. - e^{-\lambda d_X(id, h)} \mathbb{E} \left[e^{\lambda d_X(id, Z_k)} \mathbb{1}_{A^c} | Z_k^{-1} Z_m = h \right] \right) \\
&= e^{2C\lambda} \left(e^{-\lambda d_X(id, h)} \mathbb{E} \left[e^{\lambda d_X(id, Z_k)} \right] + \mathbb{E} \left[e^{\lambda d_X(id, Z_k)} \mathbb{1}_{A^c} | Z_k^{-1} Z_m = h \right] (e^{\lambda d_X(id, h)} - e^{-\lambda d_X(id, h)}) \right).
\end{aligned}$$

Using Cauchy-Schwartz and Proposition 9.4-(1) we get

$$\begin{aligned}
\mathbb{E} \left[e^{\lambda d_X(id, Z_k)} \mathbb{1}_{A^c} | Z_k^{-1} Z_m = h \right] &\leq \mathbb{E} \left[e^{2\lambda d_X(id, Z_k)} \right]^{1/2} \mathbb{P} \left[A^c | Z_k^{-1} Z_m = h \right]^{1/2} \\
&\leq \sqrt{1/10} \mathbb{E} \left[e^{2\lambda d_X(id, Z_k)} \right]^{1/2}.
\end{aligned}$$

Using this and integrating with respect to h we get

$$\begin{aligned}
\mathbb{E} \left[e^{-\lambda(d_X(id, gZ_m) - d_X(id, g))} \right] &\leq e^{2C\lambda} \mathbb{E} \left[e^{-\lambda d_X(id, Z_{m-k})} \right] \mathbb{E} \left[e^{\lambda d_X(id, Z_k)} \right] \\
&\quad + e^{2C\lambda} \mathbb{E} \left[e^{\lambda d_X(id, Z_{m-k})} - e^{-\lambda d_X(id, Z_{m-k})} \right] \sqrt{1/10} \mathbb{E} \left[e^{2\lambda d_X(id, Z_k)} \right]^{1/2} := \phi(\lambda).
\end{aligned}$$

Notice that ϕ does not depend on g and $\phi(0) = 1$. Also,

$$\phi'(0) = 2C - \mathbb{E}[d_X(id, Z_{m-k})] + \mathbb{E}[d_X(id, Z_k)] + 2\sqrt{1/10} \mathbb{E}[d_X(id, Z_{m-k})].$$

Hence, in view of Proposition 9.4-(2), we can choose m so that $\phi'(0) < 0$, and the Claim follows. \square

Let us now fix λ, ϵ, m as in the Claim, and any $g_0 \in G$. For a positive integer j , write

$$d_X(id, g_0 Z_{jm+m}) - d_X(id, g_0) = (d_X(id, g_0 Z_{jm+m}) - d_X(id, g_0 Z_{jm})) + d_X(id, g_0 Z_{jm}) - d_X(id, g_0).$$

By the Claim we have, for each $g \in G$,

$$\mathbb{E}[e^{-\lambda(d_X(id, g_0 Z_{jm+m}) - d_X(id, g_0 Z_{jm}))} | Z_{jm} = g] \leq (1 - \epsilon).$$

So, summing over all possible g ,

$$\mathbb{E}[e^{-\lambda(d_X(id, Z_{jm+m}) - d_X(id, g_0))}] \leq (1 - \epsilon) \mathbb{E}[e^{-\lambda(d_X(id, g_0 Z_{jm}) - d_X(id, g_0))}],$$

and inductively we get

$$\mathbb{E}[e^{-\lambda(d_X(id, g_0 Z_{jm}) - d_X(id, g_0))}] \leq (1 - \epsilon)^j.$$

Using Markov's inequality, for any $c > 0$ we can make the estimate

$$\mathbb{P}[d_X(id, g_0 Z_{jm}) - d_X(id, g_0) < cjm] = \mathbb{P}[e^{-\lambda(d_X(id, Z_{jm}) - d_X(id, g_0))} > e^{-\lambda cjm}] \leq e^{\lambda cjm} (1 - \epsilon)^j.$$

Choosing c small enough, we see that there exists $C_0 \geq 1$ so that

$$\mathbb{P}[d_X(id, g_0 Z_{jm}) - d_X(id, g_0) < jm/C_0] \leq e^{-jm/C_0}. \quad (*)$$

If n is now any positive integer, we can write $n = jm + r$, with $0 \leq r < m$.

Since $d_X(id, g_0 Z_n) - d_X(id, g_0) \geq d_X(id, g_0 Z_{jm}) - d_X(g_0 Z_{jm}, g_0 Z_n) - d_X(id, g_0)$, we can make the estimate

$$\begin{aligned}
&\mathbb{P}[d_X(id, g_0 Z_n) - d_X(id, g_0) < n/(2C_0)] \leq \\
&\mathbb{P}[d_X(id, g_0 Z_{jm}) - d_X(id, g_0) < jm/C_0] + \max_{i=0, \dots, m-1} \mathbb{P}[d_X(id, Z_i) \geq (jm - i)/(2C_0)].
\end{aligned}$$

The first term decays exponentially in j , whence in n , because of $(*)$, while the exponential decay of the second term follows from the exponential tail of μ_0 .

This concludes the proof. \square

10 Deviation from quasi-geodesics

Definition 10.1. Let G be a finitely generated group acting acylindrically on the geodesic hyperbolic space X . The geodesic metric space Y endowed with an isometric group action of G is acylindrically intermediate for (G, X) if there exists a G -equivariant map $\pi : Y \rightarrow X$ so that Lemma 8.1 holds for π , namely there exist $x_0 \in X$, $L \geq 0$ and a non-decreasing function f so that for each $l_1, l_2 \geq 0$, each $t \geq 0$ and whenever $x, y \in Gx_0$ satisfy $d_X(x, y) \geq L + l_1 + l_2$, we have

$$\text{diam}^Y(\pi^{-1}(B^X(x, l_1)) \cap N_t^Y(\pi^{-1}(B^X(y, l_2))) \leq f(t),$$

where diam^Y and N_t^Y denote the diameter and neighborhood taken with respect to the metric of Y .

Proposition 10.2. The following are examples of groups G and metric spaces X, Y so that Y is acylindrically intermediate for (G, X) .

1. If G is a finitely generated group acting acylindrically on the geodesic hyperbolic space X then $Y = X$ and $Y = \text{Cay}(G, S)$, for S a finite generating set of G , are both acylindrically intermediate for (G, X) .
2. If G is relatively hyperbolic, X is its coned-off graph and Y is its Bowditch space¹, then Y is acylindrically intermediate for (G, X) . Similarly, if G is the fundamental group of a finite-volume hyperbolic n -manifold, then we can take $Y = \mathbb{H}^n$.
3. If G is the mapping class group of a connected oriented hyperbolic surface S of finite type, X is the curve complex of S and Y is Teichmüller space of the surface endowed with either the Teichmüller or the Weil-Petersson metric, then Y is acylindrically intermediate for (G, X) .

Proof. Item 1) with $Y = X$ is obvious, while with $Y = \text{Cay}(G, S)$ it follows from Lemma 8.1.

2) The natural map π from Y to X maps geodesics within bounded Hausdorff distance of quasi-geodesics, see e.g [Bow12, Lemma 7.3]. In particular, since Y is hyperbolic, there exists C so that for any ball B in X there exists a C -quasiconvex set $Q(B)$ in Y (the union of all geodesics connecting points in $\pi^{-1}(B)$) so that $\pi^{-1}(B) \subseteq Q(B) \subseteq \pi^{-1}(N_C^X(B))$. If B_1 and B_2 are far away balls in X , then $Q(B_1)$ and $Q(B_2)$ are far away in Y , and hence in this case $Q(B_1) \cap N_t^Y(Q(B_2))$ can be bounded in terms of t, C and the hyperbolicity constant of Y .

If G is the fundamental group of a finite-volume hyperbolic n -manifold, then \mathbb{H}^n is equivariantly quasi-isometric to the Bowditch space for $\pi_1(M)$ (with respect to the standard relatively hyperbolic structure).

3) The acylindricity of the action of G on X was proven in [Bow08]. The statement about Teichmüller space is a consequence of the distance formula and related machinery for the Weil-Petersson [MM99, MM00, Bro03] and Teichmüller metric [Raf07, Dur]. The proofs for the two metrics are pretty much identical. The proof below gives a rather rough bound, but we tried to keep it simple as possible.

For every (isotopy class of essential) subsurface Z of S , there is an associated hyperbolic metric space C_Z and a map $\pi_Z : Y \rightarrow C_Z$. (More specifically, if Z not an annulus, then C_Z is the curve complex of Z . If Z is an annulus and we are considering the Teichmüller metric, C_Z is quasi-isometric to a horoball in \mathbb{H}^2 , while if we are considering the Weil-Petersson metric we can take Z to be a point.)

The distance formula says the following. For A, R real numbers, denote $[A]_L = A$ if $A \geq L$ and 0 otherwise. Also, write $A \approx_K B$ if $A/K - K \leq B \leq KA + K$, and similarly for \lesssim . Then for each large enough L there exists K so that for each $x, y \in Y$ we have

$$d_Y(x, y) \approx_K \sum_Z [d_{C_Z}(\pi_Z(x), \pi_Z(y))]_L,$$

¹By Bowditch space we mean the space obtained attaching combinatorial horoballs to parabolic subgroups as in [Bow12, GM08]

where the sum is taken over all (isotopy classes of essential) subsurfaces Z of S .

We will also need the consequence of the bounded geodesic image theorem that says that there exists C so that, for $x, y, x', y' \in Y$, if the geodesics from $\pi_S(x')$ to $\pi_S(y')$ are further than C from geodesics from $\pi_S(x)$ to $\pi_S(y)$, then whenever Z is a subsurface for which $d_{C_Z}(\pi_Z(x), \pi_Z(y)) \geq C$, we have $d_{C_Z}(\pi_Z(x'), \pi_Z(y')) \leq C$.

Now, suppose that the balls B_1, B_2 in $X = C_S$ are far enough apart and fix $t \geq 0$. Let $x, y \in \pi_S^{-1}(B_1)$ and let $x', y' \in \pi_S^{-1}(B_2)$ with $d_Y(x, x'), d_Y(y, y') \leq t$, for some t . We wish to bound $d_Y(x, y)$ in terms of t . Let $L > C$ be large enough. In the estimate below we write \approx instead of \approx_K and it is understood that K depends only on L :

$$\begin{aligned} d_Y(x, y) &\approx \sum_Z [d_{C_Z}(\pi_Z(x), \pi_Z(y))]_{3L} \lesssim \sum_Z [d_{C_Z}(\pi_Z(x), \pi_Z(x')) + C + d_{C_Z}(\pi_Z(y'), \pi_Z(y))]_{3L} \\ &\lesssim 3 \left(\sum_Z [d_{C_Z}(\pi_Z(x), \pi_Z(x'))]_L + \sum_Z [d_{C_Z}(\pi_Z(y'), \pi_Z(y))]_L \right) \lesssim t, \end{aligned}$$

as required. The third inequality follows from the fact that if $d_{C_Z}(\pi_Z(x), \pi_Z(x')) + C + d_{C_Z}(\pi_Z(y'), \pi_Z(y)) \geq 3L$ then $\max\{d_{C_Z}(\pi_Z(x), \pi_Z(x')), d_{C_Z}(\pi_Z(y'), \pi_Z(y))\} \geq L$. \square

10.1 Superlinear divergence

Convention. To save notation, when the group G acts on X and Y is acylindrically intermediate for (G, X) , we automatically fix basepoints $x_0 \in X$, $y_0 \in Y$ so that $x_0 = \pi(y_0)$, where π is as in Definition 10.1. Also, we identify G with the orbit in Y of y_0 . Also, we set, for $g, h \in G$, $d_X(g, h) = d_X(gx_0, hx_0)$.

Proposition 10.3. *Let G act acylindrically on the hyperbolic space X and let Y be acylindrically intermediate for (G, X) . Then for any L there exists a constant C and a diverging function $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ so that the following holds. Let α_1, α_2 be L -Lipschitz paths with respect to the metric of Y , where α_i connects g_i to h_i . Then*

$$\max\{l_Y(\alpha_1), l_Y(\alpha_2)\} \geq (d_X(g_1, h_1) - d_X(g_1, g_2) - d_X(h_1, h_2) - C) \cdot \rho(d_G(\alpha_1, \alpha_2)).$$

Proof. Let δ be the hyperbolicity constant of X . We denote by C_i suitable constants depending on G, X, Y, L, δ only, and for convenience we take $C_{i+1} \geq C_i$.

The first lemma ensures that we can assume that the paths α_i stay close to d_X -geodesics, for otherwise they would be long due to the geometry of X .

Lemma 10.4. *Let α be a path in G which is L -Lipschitz for the metric of Y and which connects g to h . Consider n points p_i in X within distance 2δ from a geodesic from gx_0 to hx_0 .*

Suppose that α satisfies $d_X(\alpha, p_i) \geq r$ for each i , for some $r \geq C_0$. Then $l_Y(\alpha) \geq nr^2/C_0$.

We could replace r^2 with an exponential function, but we do not need this fact.

Proof. We will consider the path β obtained projecting α to X and interpolating the “jumps” by geodesics. It suffices to prove that the length of β is bounded from below.

Notice that $d_X(\beta, p_i) \geq r - L$ for each i . Also, we can find disjoint subpaths β_i of β connecting, say, x_i to y_i so that a given geodesic from x_i to y_i contains a point q_i 10δ -close to p_i . In particular, β_i avoid $B^X(p, r - L - 10\delta)$. It is well-known that $l_X(\beta_j)$ is exponential in $r - L - 10\delta$, which easily implies the conclusion. \square

Lemma 10.5. *There exists a diverging non-decreasing function $\rho_0 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with the following properties. Let $a_i, b_i \in G$, for $i = 1, 2$ be so that*

- $d_X(a_i, b_i) \geq d_X(a_1, a_2) + d_X(b_1, b_2) + C_1$.

Then, denoting $s = \min\{d_Y(a_1, a_2), d_Y(b_1, b_2)\}$, we have

$$\max\{d_Y(a_i, b_i)\} \geq \rho_0(s).$$

Proof. Recall that we denote by diam^* the diameter, by $B^*(\cdot, R)$ a ball of radius R and by N_t^* a neighborhood of radius t , where $*$ can be either Y or X depending on which metric we are using to define the given notion. Recall that we are assuming the following: There exist C_1 and a non-decreasing function f so that for each t and whenever $g, h \in G$ satisfy $d_X(g, h) \geq r_1 + r_2 + C_1$, we have $\text{diam}^Y(B^X(g, r_1) \cap N_t^Y(B^X(h, r_2))) \leq f(t)$.

Let ρ_0 be a non-decreasing diverging function so that $f(\rho_0(t)) < t$ for each t .

If we had $d_Y(a_i, b_i) < \rho_0(s)$ for $i = 1, 2$, then we would have

$$f(\rho_0(s)) \geq \text{diam}^Y(B^X(a_1, d_X(a_1, a_2)) \cap N_{\rho_0(s)}^Y(B^X(b_1, d_X(b_1, b_2)))) \geq d_Y(a_1, a_2) \geq s > f(\rho_0(s)),$$

a contradiction. □

Let $r : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a diverging function so that $\rho_0(t)/r(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Let $K = d_X(g_1, h_1) - d_X(g_1, g_2) - d_X(h_1, h_2) - C_1$. We can assume $K \geq 0$. We let $r = r(d_Y(\alpha_1, \alpha_2))$. We can and will assume $r > 2C_0$.

If we have $K \leq 6r + 2C_1$ then we can make the following estimate. By Lemma 10.5 above, there exists i so that $d_Y(g_i, h_i) \geq \rho_0(d_Y(\alpha_1, \alpha_2))$. Hence,

$$l_Y(\alpha_i) \geq K \frac{\rho_0(d_Y(\alpha_1, \alpha_2))}{6r + 2C_1},$$

and we are done.

Suppose now $K \geq 6r + 2C_1$. Consider any geodesic γ_i in X from $g_i x_0$ to $h_i x_0$ and fix a sequence of points $\{p_k\}_{k=1, \dots, n}$ appearing in the given order along γ_1 so that

1. $n \geq K/(6r + 2C_1)$,
2. $d_X(p_k, p_{k+1}) \geq 3r + C_1$,
3. $d_X(p_k, \gamma_2) \leq 2\delta$.

If for some i , α_i avoids $B_{r/2}^X(p_k)$ for at least $n/10$ values of k , then, in view of Lemma 10.4, we have

$$l_Y(\alpha_i) \geq \frac{nr^2}{40C_0} \geq K \frac{r^2}{40C_0(6r + 2C_1)},$$

and we are done.

Otherwise, there are more than $4n/5$ values of k so that α_i contains a point q_k^i in $B_r^X(p_k)$ for $i = 1, 2$. We also assume that the points q_k^i appear in the given order along α_i and we set $q_0^i = g_i$, $q_{n+1}^i = h_i$. By Lemma 10.5 above, there exists i so that for more than $2n/5$ values of k we have $d_Y(q_k^i, q_{k+1}^i) \geq \rho_0(d_Y(\alpha_1, \alpha_2))$. But then

$$l_Y(\alpha_i) \geq \sum d_Y(q_k^i, q_{k+1}^i) \geq \frac{2n}{5} \rho_0(d_Y(\alpha_1, \alpha_2)) \geq K \frac{2\rho_0(d_Y(\alpha_1, \alpha_2))}{5(6r + 2C_1)},$$

as required. □

10.2 Main argument

Theorem 10.6. *Let G be a finitely generated group acting acylindrically on the geodesic hyperbolic space X , and let Y be acylindrically intermediate for (G, X) . Let μ_0 be a measure on G with exponential tail whose support generates a non-elementary group that acts with unbounded orbits on X . Then μ_0 has a neighborhood \mathcal{N} so that for every D there exists C with the following property. For each $\mu \in \mathcal{N}$, $l, n \geq 1$ and $k < n$ we have*

$$\mathbb{P}^\mu \left[\sup_{\alpha \in QG_D(id, Z_n)} d_Y(Z_k, \alpha) \geq l \right] \leq Ce^{-l/C},$$

where $QG_D(a, b)$ denotes the set of all (D, D) -quasi-geodesics (with respect to d_Y) from a to b .

First of all, we point out a corollary of the theorem. Given a metric space X , a **K -quasi-ruler** in X is a map $\gamma : I \rightarrow X$, where $I \subseteq \mathbb{R}$ is an interval, so that for all $s \leq t \leq u$ in I the Gromov product satisfies $(\gamma(s), \gamma(u))_{\gamma(t)} \leq K$.

Following [BHM11], we will say that a metric is **quasi-ruled** if there exists K so that any two points can be joined by a (K, K) -quasi-geodesic K -quasi-ruler.

Corollary 10.7. *Under the hypotheses of Theorem 10.6, let d be a quasi-ruled metric on G (e.g. a geodesic metric) quasi-isometric to d_Y . Then μ_0 satisfies the locally uniform exponential-tail deviation inequality with respect to d .*

Proof. Let D be so that d is (D, D) -quasi-isometric to d_Y , and any two points x, y of Y can be joined by a (D, D) -quasi-geodesic D -quasi-ruler $r(x, y)$. Then for each $x, y \in G$, we have $(id, y)_x \leq Dd_Y(x, r(id, y)) + 2D$ (the Gromov product is measured with respect to d). In fact, for any $p \in r(id, y)$ so that $d_Y(x, p) = d_Y(x, r(id, y))$, we have

$$(id, y)_x \leq (id, y)_p + d(x, p) \leq D + (Dd_Y(x, r(id, y)) + D).$$

Hence, the following holds for C as in Theorem 10.6. For all $n \geq k \geq 1$, $\mu \in \mathcal{N}$ and $l > 3D$ we have

$$\mathbb{P}^\mu[(id, Z_n)_{Z_k} \geq l] \leq \mathbb{P}^\mu \left[d_Y(Z_k, r(id, Z_n)) \geq \frac{l}{D} - 2 \right] \leq Ce^{-l/(CD) - 2/C},$$

as required. \square

Fix the notation of Theorem 10.6. We rescale the metric of X so that we have the inequality $d_X \leq d_Y$. When we write an inequality involving \mathbb{P} without explicit reference to the measure we mean that the statement holds for every $\mu \in \mathcal{N}$ and that the constants involved can be chosen uniformly for all $\mu \in \mathcal{N}$, where \mathcal{N} is a small enough neighborhood of μ_0 . Up to increasing D , we can replace $QG_D(\cdot, \cdot)$ in the statement with the family $QG'(\cdot, \cdot)$ of D -Lipschitz (D, D) -quasi-geodesics with given endpoints. Also, we will denote by $\gamma(g, h)$ any element of $QG'(g, h)$. In particular:

Remark 10.8. $l_Y(\gamma(g, h)) \leq D^2 d_Y(g, h) + D^3$ for each $g, h \in G$.

Proof of Theorem 10.6. We denote by $C_i \geq 1$ suitable constants that do not depend on k, n .

The fact that μ_0 has exponential tail implies that

$$\mathbb{P}[l_Y((Z_i)_{i \leq n}) \geq C_0 n] \leq C_0 e^{-n/C_0} \quad (*)$$

for a suitable C_0 .

Recall that the following holds.

Theorem 10.9. (Theorem 9.1) (Z_n) makes linear progress with exponential decay in the d_X -metric, i.e. we have

$$\mathbb{P}[d_X(id, Z_n) < n/C_1] \leq C_1 e^{-n/C_1}.$$

We say that a path $(w_i)_{i \leq n}$ is *tight around w_k at scale l* if it satisfies the following conditions for any $k_1 \leq k \leq k_2$ with $k_2 - k_1 \geq l$.

1. $d_X(w_{k_1}, w_{k_2}) \geq (k_2 - k_1)/C_1$,
2. $l_Y((w_i)_{k_1 \leq i \leq k_2}) \leq C_0(k_2 - k_1)$,
3. $d_Y(w_{k'}, w_{k'+1}) \leq \max\{l, |k - k'|/(100C_1)\}$ for each k' .

The third item says that the geodesic connecting the endpoints of the jump at step k' has length at most l if $|k' - k|$ is small and at most $|k' - k|/(100C_1)$ if $|k' - k|$ is large (recall that k is fixed and l is a parameter).

Lemma 10.10. *There exists C_5 so that for all k and all $l \geq 1$ we have*

$$\mathbb{P}[(Z_k)_{k \leq n} \text{ is tight around } Z_k \text{ at scale } l] \geq 1 - C_5 e^{-l/C_5}.$$

Proof. The probability that 1) does not hold for given k_1, k_2 can be estimated using Theorem 10.9. In fact, we have

$$\begin{aligned} \mathbb{P}[d_X(Z_{k_1}, Z_{k_2}) < (k_2 - k_1)/C_1] &= \\ \mathbb{P}[d_X(id, Z_{k_2 - k_1}) < (k_2 - k_1)/C_1] &\leq C_1 e^{-(k_2 - k_1)/C_1} \end{aligned}$$

since the law of $Z_{k_1}^{-1} Z_{k_2}$ is the same as the law of $Z_{k_2 - k_1}$.

So, for a given k_1 we get

$$\begin{aligned} \mathbb{P}[\exists k_2 \geq k : k_2 - k_1 \geq l, d_X(Z_{k_1}, Z_{k_2}) < (k_2 - k_1)/C_1] &\leq \\ \sum_{k_2 - k_1 = j \geq \max\{l, k - k_1\}} C_1 e^{-j/C_1} &\leq C_3 e^{-\max\{l, k - k_1\}/C_1}. \end{aligned}$$

Summing again over all possible k_1 we get:

$$\begin{aligned} \mathbb{P}[\exists k_1 \leq k \leq k_2 : k_2 - k_1 \geq l, d_X(Z_{k_1}, Z_{k_2}) < (k_2 - k_1)/C_1] &\leq \\ \sum_{\substack{k_1 \leq k \\ k - k_1 \leq l}} C_3 e^{-l/C_1} + \sum_{k - k_1 = j > l} C_3 e^{-j/C_1} &\leq \\ C_3 l e^{-l/C_1} + C_4 e^{-l/C_1} &\leq C_5 e^{-l/C_5}, \end{aligned}$$

what we wanted.

Items 2) and 3) can be obtained using the same summing procedure as item 1), we will not spell out the details. In the case of item 2) one uses (*), while in the case of item 3) one uses that $\mathbb{P}[d_Y(id, Z_1) \geq l]$ decays exponentially in l . \square

We now reduced the proof of Theorem 10.6 to the following entirely geometric lemma.

Lemma 10.11. *Let $(w_i)_{0 \leq i \leq n}$ be tight around w_k at scale l . Then $d_Y(w_k, \gamma(w_0, w_n)) \leq C_7 l$.*

Proof. For convenience, set $\gamma = \gamma(w_0, w_n)$. Recall that we are assuming $d_X \leq d_Y$.

We now choose some constants. Let ρ be as in Proposition 10.3, where $K = 6C_0C_1$ and $L = D$, and fix C_6 so that $\rho(t) > \max\{2C_0C_1, 4D^2C_0C_1 + 2C_0D^3\}$ for each $t \geq C_6$. Up to increasing C_6 , we can also require $C_6 \geq C$, where C is as in Proposition 10.3.

Suppose $d_Y(w_k, \gamma) \geq C_6l$, for otherwise we are done. Let $k_1 < k$ be maximal (resp. $k_2 \geq k$ be minimal) so that $\gamma(w_{k_1-1}, w_{k_1})$ (resp. $\gamma(w_{k_2}, w_{k_2+1})$) intersects the neighborhood $N_{C_6l}(\gamma)$ in Y .

Let α be the concatenation of $\gamma(w_i, w_{i+1})$ for $k_1 \leq i \leq k_2 - 1$. In particular, $d_Y(\alpha, \gamma) \geq C_6l$, and, denoting $[w_{k_i}, w_{k_i \pm 1}]$ any geodesic in Y from w_{k_i} to $w_{k_i \pm 1}$, we have

$$\begin{aligned} d_Y(w_{k_i}, \gamma) &\leq d_Y(w_{k_i}, w_{k_i \pm 1}) + d_Y([w_{k_i}, w_{k_i \pm 1}], \gamma) \\ &\leq \max\{l, (k_2 - k_1)/(100C_1)\} + C_6l. \end{aligned}$$

Also, by property 2) from the definition of tightness, we have $l_Y(\alpha) \leq C_0(k_2 - k_1)$.

We analyse 2 cases, with the aim of showing that only the first one can hold.

The first case is if $k_2 - k_1 \leq 100C_6C_1l$. Then

$$d_Y(w_k, \gamma) \leq d_Y(w_k, w_{k_1}) + d_Y(w_{k_1}, \gamma) \leq C_0(k_2 - k_1) + C_6l + \max\{l, (k_2 - k_1)/(100C_1)\},$$

which is bounded linearly in l .

The second case is if $k_2 - k_1 \geq 100C_6C_1l$. (Recall that we have to show that this does not happen.) In this case $\max\{l, (k_2 - k_1)/(100C_1)\} = (k_2 - k_1)/(100C_1)$. Let $x_{k_i} \in \gamma$ be so that $d_Y(x_{k_i}, w_{k_i}) \leq C_6l + (k_2 - k_1)/(100C_1)$, and let γ' be the subpath of γ connecting x_{k_1} to x_{k_2} . We remark that

$$d_X(x_{k_i}, w_{k_i}) \leq C_6l + \frac{k_2 - k_1}{100C_1} \leq \frac{k_2 - k_1}{4C_1} - C.$$

Hence, we have

$$K = d_X(w_{k_1}, w_{k_2}) - d_X(x_{k_1}, w_{k_1}) - d_X(x_{k_2}, w_{k_2}) - C \geq (k_2 - k_1)/(2C_1).$$

We also have $d_Y(x_{k_1}, x_{k_2}) \leq 2C_0(k_2 - k_1)$, so that

$$l_Y(\gamma') \leq 2D^2C_0(k_2 - k_1) + D^3 < \frac{k_2 - k_1}{2C_1} \rho(C_6l) \leq K\rho(C_6l).$$

Hence, by Proposition 10.3 we have

$$k_2 - k_1 \geq \frac{l_Y(\alpha)}{C_0} \geq \frac{K\rho(C_6l)}{C_0} \geq \frac{k_2 - k_1}{2C_0C_1} \rho(C_6l) > k_2 - k_1,$$

a contradiction. So the second case cannot hold and the proof is complete. \square

11 Deviation inequalities in hyperbolic groups

In this section we study deviation inequalities in hyperbolic groups under very general conditions on driving measures. Perhaps surprisingly at first, we will see that driving measures μ with finite second moment satisfy the p -th moment deviation inequality for each $p < 4$. As we will see in the proofs, this follows from the fact that, roughly speaking, the way that a sample path can deviate from a geodesic is by doing two large (almost) consecutive steps. Hence, once again roughly speaking, the probability that a point on a sample path is far from a geodesic connecting the endpoints of the path is comparable to that of having two large consecutive jumps.

Lemma 11.4 below is where we exploit the geometry of hyperbolic groups. In the given form, the lemma does not hold for acylindrically hyperbolic groups, and it is unclear whether a useful version of the lemma exists in that context.

Theorem 11.1. *Let G be a hyperbolic group endowed with the word metric d_G and let μ_0 be a measure with finite first moment on G whose support generates a non-elementary subgroup of G . Then there exists a neighborhood \mathcal{N} of μ_0 and a constant $C \geq 1$ so that for every $\mu \in \mathcal{N}$ the random walk (Z_n) with driving measure μ satisfies the following. For each $t \geq 0$, each $M \geq 1$ and each positive integers $k \leq n$ we have*

$$\mathbb{P}^\mu \left[\sup_{[id, Z_n]} d_G(Z_k, [id, Z_n]) \geq t \right] \leq C e^{-M/C} + M^2 (\mathbb{P}^\mu [d(id, X_1) \geq (t - C)/M])^2.$$

Moreover, if μ has finite second moment then μ satisfies the p -th moment deviation inequality for each $p < 4$.

Fix the notation of the theorem from now on. The constants C_i appearing below depend on the data of the theorem and are all uniform in a sufficiently small neighborhood of μ_0 .

Lemma 11.2. *There exists C_0 with the following property. For each $n \geq 1$ and each $g, h \in G$ we have*

$$\mathbb{P}^\mu [d_G(id, gZ_n h) \leq n/C_0] \leq C_0 e^{-n/C_0}.$$

Proof. Since non-elementary subgroups of G are non-amenable, there exists K so that for each $a \in G$ we have $\mathbb{P}^\mu [Z_n = a] \leq K e^{-n/K}$ [Woe00]. If $\{a_i\}$ are the elements in the ball of radius n/C_0 in G then

$$\mathbb{P}^\mu [d_G(id, gZ_n h) \leq n/C_0] = \sum_i \mathbb{P}^\mu [Z_n = g^{-1} a_i h^{-1}] \leq |B^G(id, n/C_0)| K e^{-n/K},$$

and the conclusion easily follows for C_0 large enough. \square

We fix, for each $x, y \in G$, a geodesic $[x, y]$ connecting them.

Lemma 11.3. *There exists C_1 with the following property. For each $n \geq 1$ we have*

$$\mathbb{P}^\mu [d_G([id, Z_1], [Z_n, Z_{n+1}]) \leq n/C_1] \leq C_1 e^{-n/C_1}.$$

Proof. For C_0 as in Lemma 11.2 we have

$$\begin{aligned} & \mathbb{P}^\mu [d_G([1, Z_1], [Z_n, Z_{n+1}]) \leq n/C_0] \\ &= \sum_{h_1, h_2 \in G} \sum_{x_i \in [id, h_i]} \mathbb{P}^\mu [d_G(x_1, h_1 Z_{n-1} x_2) \leq n/C_0] \mathbb{P}^\mu [X_1 = h_1, X_{n+1} = h_2] \\ &\leq C_0 e^{-n/C_0} \sum_{h_1, h_2 \in G} d_G(id, h_1) d_G(id, h_2) \mathbb{P}^\mu [X_1 = h_1, X_{n+1} = h_2] = C_0 e^{-n/C_0} \mathbb{E}^\mu [d(id, X_1)]^2, \end{aligned}$$

as required. \square

Lemma 11.4. *There exists C_2 with the following property. Let γ be a geodesic in G and let $(w_i)_{i=0, \dots, n}$ be a discrete path with endpoints on γ . Let $M \geq C_2$ be a positive integer. Then for each $k \in \{0, \dots, n\}$ one of the following holds.*

1. *There exist $k_1 < k \leq k_2$ with $|k_2 - k_1| \leq M$ so that $d_G(w_{k_1}, w_{k_1+1}), d_G(w_{k_2}, w_{k_2+1}) \geq (d_G(w_k, \gamma) - C_2)/M$.*
2. *There exist $k_1 < k \leq k_2$ with $|k_2 - k_1| \geq M$ so that $d_G([w_{k_1}, w_{k_1+1}], [w_{k_2}, w_{k_2+1}]) \leq (k_2 - k_1)/C_2$.*

3. There exist $k_1 < k \leq k_2$ with $|k_2 - k_1| \geq M$ so that $\sum_{k_1 \leq i < k_2} d_G(w_i, w_{i+1}) \geq e^{(k_2 - k_1)/C_2}/C_2$.

Proof. We fix a choice $\pi_\gamma : G \rightarrow \gamma$ of closest point projection onto γ .

Denote $N^\bowtie(\gamma)$ be the closure of the set of all $x \in G$ so that $d_G(x, \pi_\gamma(x)) \leq d_G(\pi_\gamma(w_k), \pi_\gamma(x))$.

Let $k_1 < k$ be maximal (resp. $k_2 \geq k$ be minimal) so that $[w_{k_1}, w_{k_1+1}] \cap N^\bowtie(\gamma) \neq \emptyset$ (resp. $[w_{k_2}, w_{k_2+1}] \cap N^\bowtie(\gamma) \neq \emptyset$).

Claim 1. $d_G(w_{k_i}, w_k) \geq d_G(w_k, \gamma) - 100\delta$.

Proof of Claim 1. If $d_G(\pi_\gamma(w_k), \pi_\gamma(w_{k_i})) \leq 10\delta$ then

$$d_G(w_k, \gamma) = d_G(w_k, \pi_\gamma(w_k)) \leq d_G(w_k, w_{k_i}) + d_G(w_{k_i}, \pi_\gamma(w_{k_i})) + d_G(\pi_\gamma(w_{k_i}), \pi_\gamma(w_k)) \leq d_G(w_k, w_{k_i}) + 20\delta.$$

If $d_G(\pi_\gamma(w_k), \pi_\gamma(w_{k_i})) \geq 10\delta$ then any geodesic from w_k to w_{k_i} passes 10δ -close to $\pi_\gamma(w_k)$ and $\pi_\gamma(w_{k_i})$. Hence

$$d_G(w_k, w_{k_i}) \geq d_G(w_k, \pi_\gamma(w_k)) + d_G(\pi_\gamma(w_k), \pi_\gamma(w_{k_i})) + d_G(\pi_\gamma(w_{k_i}), w_{k_i}) - 100\delta \geq d_G(w_k, \gamma) - 100\delta.$$

□

In view of Claim 1, if we have $|k_1 - k| \leq M$ and $|k_2 - k| \leq M$ then 1) holds, possibly for different k_1, k_2 . In fact, $\sum_{k'=k_1, \dots, k-1} d_G(w_{k'}, w_{k'+1}) \geq d_G(w_{k_1}, w_k) \geq d_G(w_k, \gamma) - 100\delta$, so that one of the terms of the sum is large, and similarly on the “other side” of k .

Hence, suppose that either $|k_1 - k| > M$ or $|k_2 - k| > M$, so that in particular $|k_2 - k_1| \geq M$. Also, suppose that 2) does not hold, for the given k_1, k_2 .

Let β be the concatenation of a subpath of $[w_{k_1}, w_{k_1+1}]$, $[w_i, w_{i+1}]$ for $i = k_1 + 1, \dots, k_2 - 1$ and a subpath of $[w_{k_2}, w_{k_2+1}]$ with the property that β intersects $N^\bowtie(\gamma)$ only at its endpoints x, y .

Claim 2. $l_G(\beta) \geq e^{d(x,y)/C_2}/C_2$.

Proof of Claim 2. If we had $d_G(\pi_\gamma(x), \pi_\gamma(w_k)), d_G(\pi_\gamma(y), \pi_\gamma(w_k)) \leq (k_2 - k_1)/(5C_0)$ then we would have

$$d_G(x, y) \leq d_G(x, \pi_\gamma(x)) + d_G(\pi_\gamma(x), \pi_\gamma(y)) + d_G(\pi_\gamma(y), y) \leq \frac{4}{5C_0}(k_2 - k_1),$$

where we used the definition of $N^\bowtie(\gamma)$. This contradicts the assumption that 2) does not hold.

Hence, let us say $d_G(\pi_\gamma(x), \pi_\gamma(w_k)) \geq (k_2 - k_1)/(5C_0)$, the other case being symmetric. The subpath β' of β from x to w_k avoids $B(\pi_\gamma(x), (k_2 - k_1)/(10C_0))$, and $\pi_\gamma(x)$ lies 10δ -close to a geodesic from x to w_k . Hence, the length of β' is exponential in $k_2 - k_1$, as required. □

In view of Claim 2, condition 3) holds, and the proof of the lemma is complete. □

Proof of Theorem 11.1. Let $M \geq C_2$ be a positive integer (for $M \leq C_2$ the theorem trivially holds setting $C = C_2$). We claim that there exists C_3 so that with probability at least $1 - C_3 e^{-M/C_3}$ a sample path (w_i) of (Z_n) does not satisfy item 2) or 3) in Lemma 11.4.

For given k_1, k_2 , the probability that 2) holds is exponentially small in $k_2 - k_1$ by Lemma 11.3, and it is easily seen that the probability that 3) holds is also exponentially small in $k_2 - k_1$. In fact, the inequality $\sum_{k_1 \leq i < k_2} d_G(w_i, w_{i+1}) \geq e^{(k_2 - k_1)/C_2}/C_2$ forces one of the summands to be exponentially large in $k_2 - k_1$, but the probability of the existence of such a jump is exponentially small by Markov's inequality.

The claim now follows summing over all possible k_1, k_2 , similarly to Lemma 10.10.

In view of Lemma 11.4, we then have

$$\mathbb{P}^\mu[d_G(Z_k, [id, Z_n]) \geq t]$$

$$\begin{aligned}
&\leq C_3 e^{-M/C_3} + \sum_{\substack{k_1 < k \leq k_2, \\ |k_2 - k_1| \leq M}} \mathbb{P}^\mu[d_G(w_{k_1}, w_{k_1+1}) \geq (t - C_2)/M] \mathbb{P}^\mu[d_G(w_{k_2}, w_{k_2+1}) \geq (t - C_2)/M] \\
&\leq C_3 e^{-M/C_3} + M^2 (\mathbb{P}^\mu[d(id, X_1) \geq (t - C_2)/M])^2,
\end{aligned}$$

as required.

For the “moreover” part, we take M of order $\log(t)$ in the expression above. The first term can then be made of order t^{-4} , while the second term, using Chebyshev, is of order $\log(t)^6/t^4$. Hence,

$$\mathbb{P}^\mu[d_G(Z_k, [id, Z_n]) \geq t] \leq C_4 \frac{\log(t)}{t^4}$$

for each large enough t and a suitable constant C_4 . We can replace “ $d_G(Z_k, [id, Z_n])$ ” in the expression above by “ $|[id, Z_n]_{Z_k}|$ ” (up to modifying the constant), hence for each $0 < \epsilon < 1$, we have

$$\mathbb{E}^\mu[|[id, Z_n]_{Z_k}|^{4-\epsilon}] = (4-\epsilon) \int_0^\infty t^{3-\epsilon} \mathbb{P}^\mu[|[id, Z_n]_{Z_k}| \geq t],$$

which is finite in view of the estimate above. \square

12 Deviation for Green metrics

In this section we prove deviation inequalities for Green metrics. We say that the measure μ on the finitely generated group G has **superexponential tail** if for every $\alpha > 0$ we have $\sum_{x \in G} e^{\alpha d_G(id, x)} \mu(x) < \infty$, where d_G is any word metric on G .

Theorem 12.1. *Let G be a finitely generated acting acylindrically and non-elementarily on the geodesic hyperbolic space X , endowed with the word metric d_G . Let μ_0 be a measure on G with superexponential tail whose support generates a non-elementary group that acts with unbounded orbits on X . Then μ_0 has a neighborhood \mathcal{N} with the following properties. There exists C so that for each symmetric $\mu \in \mathcal{N}$, each μ' in \mathcal{N} , $0 \leq k \leq n$ and $l \geq 1$ we have*

$$\mathbb{P}^{\mu'} \left[(id, Z_n)_{Z_k}^{\mu} \geq l \right] \leq C e^{-l/C},$$

where $(x, y)_w^{\mu}$ denotes the Gromov product in the Green metric d_G^μ with respect to μ .

The rest of this section is devoted to the proof of the theorem.

Fix the notation of the theorem from now on. When we write an inequality involving \mathbb{P} , d_G without explicit reference to the measure we mean that the statement holds for every $\mu \in \mathcal{N}$ and that the constants involved can be chosen uniformly for all $\mu \in \mathcal{N}$, where \mathcal{N} is a small enough neighborhood of μ_0 . We denote by $\gamma(g, h)$ any geodesic in G from g to h . Up to rescaling the metric of X , we can and will assume $d_X(g, h) \leq d_G(g, h)$ for all $g, h \in G$.

We denote by C_i suitable constants depending on the data of the theorem, and for convenience we take $C_{i+1} \geq C_i$.

A (T, S) -linear progress point $p \in \gamma(g, h)$ is a point that satisfies the following property. For each $q_1, q_2 \in \gamma(g, h)$ with $d_G(p, q_i) \geq S$ and so that q_1, p, q_2 appear in this order along $\gamma(g, h)$, we have $d_G(q_1, q_2) \leq T d_X(q_1, q_2)$.

Denote by $\gamma(g, h)_{T, S}$ the collection of all (T, S) -linear progress points $p \in \gamma(g, h)$.

The theorem follows combining the two lemmas below. We remark that the measure μ' only plays a role in Lemma 12.2, while the measure μ only plays a role in Lemma 12.3.

Lemma 12.2. *There exists T and C_5 so that for each $k, n \geq k$ and $l \geq 1$,*

$$\mathbb{P}[d_G(Z_k, \gamma(id, Z_n)_{T, C_5 l}) \geq l] \leq C_5 e^{-l/C_5}$$

The idea is that points along a random path make linear progress in d_X and stay d_G -close to $\gamma(id, Z_n)$, hence random points along $\gamma(id, Z_n)$ are of linear progress.

Proof. From Theorem 10.6 and Theorem 9.1 we know for each $k', k_1, k_2 \leq n, l' \geq 1$:

$$\mathbb{P}[d_G(Z_{k'}, \gamma(id, Z_n)) \geq l'] \leq C_1 e^{-l'/C_1}$$

and

$$\mathbb{P}[d_X(Z_{k_1}, Z_{k_2}) \leq |k_1 - k_2|/C_2] \leq C_2 e^{-|k_1 - k_2|/C_2}.$$

Also, as μ has exponential tail we have:

$$\mathbb{P}[d_G(Z_k, Z_{k'}) \geq C_3 |k' - k|] \leq C_3 e^{-|k' - k|/C_3}.$$

Let \mathcal{I} be the set of integers $k + i10C_2l \in \{0, \dots, n\}$. Summing over all $k', k_1, k_2 \leq n$ of the form $k' = k + i10C_2l$ and with $l' = l + il$, we get that the probability that (a), (b), (c) hold for each $i, i_1, i_2 \in \mathcal{I}$ with $i_1 \leq 0 \leq i_2$ is at least $1 - C_4 e^{-l/C_4}$, where

- (a) $d_X(Z_{k+i_1C_2l}, Z_{k+i_2C_2l}) \geq 10(i_2 - i_1)l$,
- (b) $d_G(Z_{k+iC_2l}, \gamma(id, Z_n)) \leq l + |i|l$,
- (c) $d_G(Z_k, Z_{k+iC_2l}) \leq |i|C_4l$.

Hence, with probability at least $1 - C_4 e^{-l/C_4}$, along the geodesic from id to the endpoint of a random walk Z_n , we have points $\{p_i\}$ so that

- 1. p_0 is l -close to Z_k ,
- 2. $d_X(p_{i_1}, p_{i_2}) \geq 10(i_2 - i_1)l - 2l - (i_2 - i_1)l \geq 7(i_2 - i_1)l$ for $i_1 \leq 0 \leq i_2$ and $i_1 \neq i_2$,
- 3. $d_G(p_i, p_0) \leq |i|C_4l$.

Such properties easily imply that p_0 is of (T, C_5l) -linear progress, as required. \square

Lemma 12.3. *Let $T \geq 1$. If $\mu \in \mathcal{N}$ is symmetric, then there exists C_{14} so that if $p \in \gamma(g, h)$ is a (T, S) -linear progress point for some $S \geq 1$ then $d_G^\mu(x, p) + d_G^\mu(p, y) \leq d_G^\mu(x, y) + C_{14}S$.*

Proof. Let us call (L, μ) -path a discrete path so that all pairs of consecutive points w_i, w_{i+1} along the path satisfy $w_{i+1} = w_i s$ for some $s \in \text{supp}(\mu) \cap B^G(id, L)$. A μ -path is a $(+\infty, \mu)$ -path. The weight $W(\alpha)$ of a μ -path α of length n is the probability that a random walk of length n driven by μ follows the path α . Notice that the Green function between two points x, y is $G^\mu(x^{-1}y) = \sum_{\alpha \in \mathcal{P}(x, y)} W(\alpha)$, where $\mathcal{P}(x, y)$ is the set of all μ -paths connecting x, y . Recall from the definitions in paragraph 6.1 that $d_G^\mu(x, y) = -\log G^\mu(x^{-1}y) + \log G^\mu(id)$. Thus it will be sufficient to prove that, for points x, y and p as in the Lemma, we have

$$G^\mu(x^{-1}p)G^\mu(p^{-1}y) \geq e^{-C_{14}S} G^\mu(id)G^\mu(x^{-1}y).$$

Adjusting the value of C_{14} , we see it suffices to prove that

$$G^\mu(x^{-1}p)G^\mu(p^{-1}y) \geq e^{-C_{14}S} G^\mu(x^{-1}y).$$

We denote $\gamma = \gamma(g, h)$ for convenience.

In the first part of the proof we show that a path avoiding a d_G -ball around $p \in \gamma$ has a long subpath with certain properties. The constants $N, K > 1$ will be defined later and depend on μ only.

We choose constants in the following way. Fix C_7 so that $\rho(t) \geq \max\{4K^3T, 2NT\}$ for all $t \geq C_7$, where ρ is as in Proposition 10.3 with $Y = G$.

If C_8 is large enough then we can argue as follows. Let α be a K -Lipschitz path from g to h that avoids $B^G(p, C_8S)$. Then we claim that we can find a subpath β of α with the following properties. “To the left” and “to the right” refer to the natural order along γ .

- β does not intersect $N_{C_7S}^G(\gamma)$.
- The endpoints g', h' of β are at d_G -distance between C_7S and $C_7S + K$ from γ .
- For some $g'', h'' \in \gamma$ with $d_G(g', g'') \leq C_7S + K$ and $d_G(h', h'') \leq C_7S + K$ we have that g'' is to the left of p and h'' is to the right of p .

In fact, we can obtain β removing the first and last point from the subpath of α connecting the last point along α that is C_7S -close to a point in γ to the left of p to another suitable point along α that is C_7S -close to a point to the right of p .

Notice that we have $d_G(g', h') \geq 2C_8S - 4(C_7S + K)$ since g' and h' are outside $B^G(p, C_8S)$ and $(C_7S + K)$ -close to points on γ on opposite sides of p . Also, $d_G(g'', h'') \geq d_G(g', h') - 2(C_7S + K)$, so that we have (for C_8 large enough)

$$d_X(g'', h'') \geq \frac{d_G(g'', h'')}{T} \geq \frac{d_G(g', h') - 2(C_7S + K)}{T} \geq \frac{d_G(g', h')}{2T} + 2C_7S + 2K + C.$$

Hence, using Proposition 10.3 we get

$$\begin{aligned} \max\{l_G(\beta), d_G(g'', h'')\} &\geq (d_X(g'', h'') - 2C_7S - 2K - C)\rho(C_7S) \\ &\geq \frac{d_G(g', h')}{2T}\rho(C_7S) \\ &\geq \max\{2K^3, N\}d_G(g', h'). \end{aligned}$$

But for K, N large enough we have $d_G(g'', h'') < \max\{2K^3, N\}d_G(g', h')$, so we get

$$l_G(\beta) > \max\{2K^3, N\}d_G(g', h'). \quad (*)$$

Let K_0 be so that any point in $B^G(id, 1)$ is connected to id by a (K_0, μ) -path of length at most K_0 .

There exists $\epsilon = \epsilon(\mu, K_0)$ so that the weight of any (K_0, μ) -path of length n is at least ϵ^n . Also, there exists $\theta = \rho_\mu < 1$ so that the probability $\mathbb{P}^\mu[Z_n = a^{-1}b]$ is at most θ^n , see (6.33). We can choose θ with the additional property that the weight of the set of all paths from a to b is at most $\theta^{d_G(a, b)}$. Let $N \geq K_0$ be so that $\epsilon^{K_0}/\theta^N \geq \theta^{-2}$.

Fix some $K \geq K_0$. Let α' be any μ -path. We can form a new path α from α' by interpolating all jumps in α' of length at least K with a (K_0, μ) -path of minimal length, i.e. whenever w_i, w_{i+1} are consecutive points along α' with $d_G(w_i, w_{i+1}) \geq K$, we can insert a (K_0, μ) -path from w_i to w_{i+1} . For I the set of indices so that $d_G(w_i, w_{i+1}) \geq K$ and $l_i = d_G(w_i, w_{i+1})$, the weights satisfy

$$W(\alpha) \geq W(\alpha') \prod_{i \in I} \frac{\epsilon^{K_0 l_i}}{f(l_i)},$$

where $f(t)$ is a function going to 0 superexponentially fast as t goes to $+\infty$ that depends on μ . In particular, for K large enough (depending on f, K_0) so that $\frac{\epsilon^{K_0 t}}{f(t)} \geq 1$ for all $t \geq K$, we have that the weight of α is at least the weight of α' . We further increase K so that $\frac{\epsilon^{K_0 t}}{f(t)} \geq \theta^{-2t}$ for each $t \geq K$ and we fix it from now on.

Claim. Let P_a be the collections of all μ -paths from g to h that avoid the ball of radius $100KC_8S$ around p , and let P_t be the collection of those that intersect it. Then $W(P_a) \leq C_{12}W(P_t)$.

Proof of Claim. Let α' be a μ -path from g to h that avoids the ball of radius $100KC_8S$ around p , and let α be a (K_0, μ) -path obtained “filling in” the jumps of size larger than K with (K_0, μ) -paths of minimal length, as we did above. We call such (K_0, μ) -paths *interpolation paths*. We analyse two cases. (We could avoid analysing the first case if the measure had finite support.)

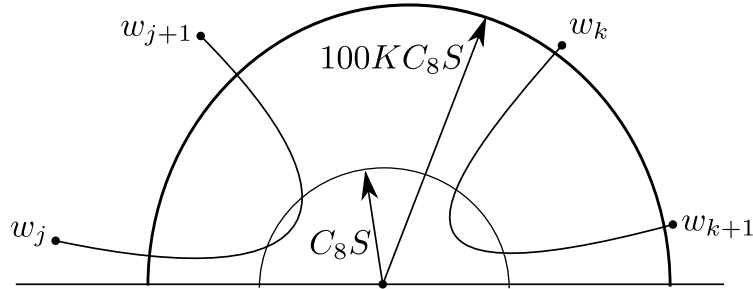
a) We want to argue that $W(Q(g, h)) \leq C_9W(P_t)$, where $Q(g, h)$ is the set of all μ -paths α' that do not intersect $B^G(p, 100KC_8S)$ but whose corresponding α intersects $B^G(p, C_8S)$. Suppose $\alpha \in Q(g, h)$.

We say that an interpolation path is *fly-by* if it intersects $B^G(p, C_8S)$. Let $\hat{\alpha}$ be obtained from α' by removing the subpath connecting the first w_j in some fly-by interpolation path to the last w_{k+1} in some fly-by interpolation path, and replacing it by a (K_0, μ) -path of minimal length. Let I be the set of indices i so that w_i, w_{i+1} are the endpoints of a fly-by interpolation path. We have

$$W(\hat{\alpha}) \geq W(\alpha') \epsilon^{K_0 d_G(w_j, w_{k+1})} \prod_{i \in I} \frac{1}{f(l_i)}.$$

We also have $d_G(w_j, w_{k+1}) \leq l_j + l_k$ (if $j \neq k$, and $d_G(w_j, w_{k+1}) = l_j$ otherwise), whence we get

$$W(\hat{\alpha}) \geq \theta^{-2d_G(w_j, w_{k+1})} W(\alpha').$$



If the map $\alpha' \mapsto \hat{\alpha}$ was 1-to-1, then this would directly give us what we want. The map is not 1-to-1, but “almost”, meaning that we can estimate the weight of the set of all α' that get mapped to a given $\hat{\alpha}$. Any such α' is obtained replacing a single (K_0, μ) -subpath of $\hat{\alpha}$ of minimal length that K_0^2 -fellow-travels a geodesic. There are boundedly many possible endpoints, say, w_j, w_{k+1} , of a “replaceable” subpath with a given $d_G(w_j, w_{k+1})$. Also, the weight of all paths connecting w_j to w_{k+1} is at most $\theta^{d_G(w_j, w_{k+1})}$. Hence, summing the inequality above yields the required estimate.

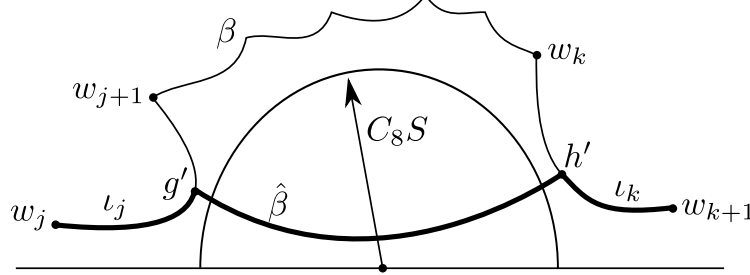
b) We now wish to show $W(P_a \setminus Q(g, h)) \leq C_{11}W(P_t)$.

Let $\alpha' \in P_a \setminus Q(g, h)$. We will construct a certain $\alpha'' \in P_t$ starting from α' , and then we will check that the map $\alpha' \mapsto \alpha''$ has the property that the weight of the preimage of a given α'' is bounded in terms of the weight of α'' .

Consider the (K_0, μ) -path α obtained interpolating α' . Let β be a subpath as in the first part of the proof, which has d_G -length at least $Nd_G(g', h')$ by (*).

Let w_j be so that g' is on the interpolation path ι_j from w_j to w_{j+1} , and let w_{k+1} be so that h' is on the interpolation path ι_k from w_k to w_{k+1} .

Let $\hat{\beta}$ be a (K_0, μ) -path obtained concatenating (in suitable order) subpaths ι'_j, ι'_k of the aforementioned interpolation paths and a (K_0, μ) -path of minimal length from g' to h' . Finally, let α'' be the concatenation of the initial subpath α'_1 of α' with final point w_j , $\hat{\beta}$ and the final subpath α'_2 of α' starting at w_{k+1} .



We showed above that adding interpolation paths does not decrease the weight. Hence, for $d_G(w_i, w_{i+1}) = l_i$, we have

$$W(\alpha') \leq W(\alpha'_1)W(\alpha'_2)f(l_j)f(l_k) \frac{W(\beta)}{W(\iota_j \setminus \iota'_j)W(\iota_k \setminus \iota'_k)}.$$

Using this estimate, we get

$$\begin{aligned} W(\alpha'') &\geq W(\alpha'_1)W(\alpha'_2)W(\iota'_j)W(\iota'_k)\epsilon^{K_0 d_G(g', h')} \\ &\geq W(\alpha') \frac{W(\iota_j)W(\iota_k)}{f(l_j)f(l_k)} \frac{\epsilon^{K_0 d_G(g', h')}}{\theta^{N d_G(g', h')}} \\ &\geq \theta^{-2(l_j + d_G(g', h') + l_k)}. \end{aligned}$$

Notice that the distance from g', h' to geodesics connecting w_j, w_{j+1} and w_k, w_{k+1} is bounded by K_0^2 , so that $l_j = d_G(w_j, w_{j+1}) \geq d_G(w_j, g') - 2K_0^2$ and similarly for l_k . Hence we conclude:

$$W(\alpha'') \geq \theta^{-2d_G(w_j, w_{k+1})} / C_{10}.$$

Once again, if the map $\alpha' \mapsto \alpha''$ was 1-1, we would be done. However, any given α' that gets mapped to α'' is obtained from α'' replacing a subpath $\hat{\beta}$ by a μ -path, say with endpoints w_j, w_{k+1} . The weight of all such paths is at most $\theta^{d_G(w_j, w_{k+1})}$. This easily implies that summing over all possible w_j, w_{k+1} yields the desired estimate.

The claim now easily follows: $W(P_a) = W(Q(g, h)) + W(P_a \setminus Q(g, h)) \leq (C_9 + C_{11})W(P_t)$. \square

Expanding the definition of the Green metric, one sees that it suffices to show the following. Let $P(a, b)$ be the collection of all μ -paths from a to b . Then

$$W(P(g, h)) \leq (C_{13})\epsilon^{-C_{13}S}W(P(g, p))W(P(p, h)).$$

It is easily seen that:

$$W(P(g, p))W(P(p, h)) \geq W(P_t)\epsilon^{2C_8 K_0 S}.$$

Hence,

$$\begin{aligned} W(P(g, h)) &= W(P_a \cup P_t) \leq (C_{12} + 1)W(P_t) \\ &\leq (C_{12} + 1)W(P(g, p))W(P(p, h))\epsilon^{-2C_8 K_0 S}, \end{aligned}$$

as required. \square

Part III

Conclusions

13 Statements of the main results

For the convenience of the reader and for future reference, in this section we state the theorems that can be obtained combining results that rely on deviation inequalities, proven in the first half of the paper, with results about getting deviation inequalities from the second part of the paper.

First of all, we collect the results about the regularity of the rate of escape. Acylindrical and non-elementary actions are defined in Section 8, while the notion of being acylindrically intermediate is given in Definition 10.1. (Recall that if the finitely generated group G acts acylindrically on the geodesic hyperbolic space X , then any Cayley graph of G and X itself are acylindrically intermediate for (G, X) , see Proposition 10.2 for more examples). Recall that we fixed the convention that, when we have a fixed action of a group G on some metric space Y , we automatically fix a basepoint $y \in Y$ and denote d_Y the metric on G defined by $d_Y(g, h) = d_Y(gy, hy)$ for each $g, h \in G$. Finally, recall that, for G a group and $B \subseteq G$, we defined $\mathcal{P}(B)$ to be the set of all measures on G supported on B and that, for d a metric on G and μ a measure on G with finite first moment, we defined the rate of escape as $\ell(\mu; d) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{x \in G} d(id, x) \mu^n(x)$. We defined a distance on $\mathcal{P}(B)$ in paragraph 5.1.

The following theorem is obtained combining Theorem 10.6 with Theorems 5.1 and 5.2.

Theorem 13.1. *Let G be a finitely generated group acting acylindrically on the geodesic hyperbolic space X , and let Y be acylindrically intermediate for (G, X) . Let μ be a measure on G with exponential tail whose support B generates a non-elementary group that acts with unbounded orbits on X . Then*

1. *there exists a neighborhood of μ in $\mathcal{P}(B)$, say \mathcal{N} , such that the function $\mu_0 \rightarrow \ell(\mu_0; d_Y)$ is Lipschitz continuous on \mathcal{N} .*
2. *the function $\mu_0 \rightarrow \ell(\mu_0; d_Y)$ is differentiable at $\mu_0 = \mu$ in the following sense: Let $(\mu_t, t \in [0, 1])$ be a curve in $\mathcal{P}(B)$ such that $\mu_0 = \mu$ and, for all $a \in B$, the function $t \rightarrow \log \mu_t(a)$ has a derivative at $t = 0$, say $\nu(a)$. We assume that ν is bounded on B and also that $\sup_{t \in [0, 1]} \sup_{a \in B} \left| \frac{1}{t} \log \frac{\mu_t(a)}{\mu_0(a)} - \nu(a) \right| < \infty$. Then the limit of $\frac{1}{t}(\ell(\mu_t; d_Y) - \ell(\mu; d_Y))$ as t tends to 0 exists. Besides this limit coincides with the covariance*

$$\sigma(\nu, \mu; d_Y) := \lim_n \frac{1}{n} \mathbb{E}^\mu[d_Y(id, Z_n) \left(\sum_{j=1}^n \nu(X_j) \right)]. \quad (13.38)$$

Here is instead the statement about the asymptotic entropy, obtained combining Theorem 12.1 with Theorems 6.1 and 6.2. Recall that, whenever G is a group and μ is a measure on G , we define the entropy $H(\mu) := \sum_{x \in G} (-\log \mu(x)) \mu(x)$ and the asymptotic entropy $h(\mu) := \lim_{n \rightarrow \infty} \frac{1}{n} H(\mu^n)$ (when $H(\mu)$ is finite). We denote $\mathcal{P}_s(B)$ the set of all symmetric measures supported on B .

Theorem 13.2. *Let G be a finitely generated group acting acylindrically and non-elementarily on the geodesic hyperbolic space X , endowed with the word metric d_G . Let μ be a measure on G with superexponential tail whose support B generates a non-elementary group that acts with unbounded orbits on X . Then*

1. *there exists a neighborhood of μ in $\mathcal{P}_s(B)$, say \mathcal{N} , such that the function $\mu_0 \rightarrow h(\mu_0)$ is Lipschitz continuous on \mathcal{N} ,*

2. the function $\mu_0 \rightarrow h(\mu_0)$ is differentiable at μ_0 in the following sense: Let $(\mu_t, t \in [0, 1])$ be a curve in $\mathcal{P}(B)$ such that $\mu_0 = \mu$ and, for all $a \in B$, the function $t \rightarrow \log \mu_t(a)$ has a derivative at $t = 0$, say $\nu(a)$. We assume that ν is bounded on B and also that $\sup_{t \in [0, 1]} \sup_{a \in B} |\frac{1}{t} \log \frac{\mu_t(a)}{\mu_0(a)} - \nu(a)| < \infty$. Then the limit of $\frac{1}{t}(h\mu_t) - h(\mu)$ as t tends to 0 exists. Besides this limit coincides with the covariance

$$\sigma_G(\nu, \mu) := \lim_n \frac{1}{n} \mathbb{E}^\mu[d_G^\mu(id, Z_n) \left(\sum_{j=1}^n \nu(X_j) \right)]. \quad (13.39)$$

We now proceed with the bound on (higher) moments of measures, obtained combining Theorem 10.6 with Theorem 4.8. Recall that we denote by τ_p the constant appearing in definition of p -th-moment deviation inequality, see Section 3.3.

Theorem 13.3. *Let G be a finitely generated group acting acylindrically on the geodesic hyperbolic space X , and let Y be acylindrically intermediate for (G, X) . Let μ be a measure on G with exponential tail whose support generates a non-elementary group that acts with unbounded orbits on X . Then for all $p > 1$, there exists a constant $c(p)$ such that*

$$\mathbb{E}^\mu[|d_Y(id, Z_n) - \mathbb{E}^\mu[d_Y(id, Z_n)]|^p] \leq c(p)(\mathbb{E}^\mu[d_Y(id, Z_1)^p] + \tau_p(\mu)) n^{p/2}.$$

Finally, we conclude with the Central Limit Theorem. Items (1) and (3) are obtained combining Theorem 10.6 with Theorems 4.1 and 4.2. Item (2) follows from Theorem 4.11 in view of the fact that, in the notation set below, we have $\ell(\mu; d_X) > 0$ by Theorem 9.1, and whence a fortiori we also have $\ell(\mu; d_Y) > 0$.

Theorem 13.4. *[Central Limit Theorem] Let G be a finitely generated group acting acylindrically on the geodesic hyperbolic space X , and let Y be acylindrically intermediate for (G, X) . Let μ be a measure on G with exponential tail whose support generates a non-elementary group that acts with unbounded orbits on X . Then the following hold.*

1. $\frac{1}{n} \mathbb{V}^\mu(d_Y(id, Z_n))$ has a limit as n tends to ∞ , which we denote by σ^2 .
2. $\sigma^2 > 0$.
3. The law of $\frac{1}{\sqrt{n}}(d_Y(id, Z_n) - \ell(\mu; d_Y)n)$ under \mathbb{P}^μ weakly converges to the Gaussian law with zero mean and variance σ^2 .

References

- [AMS13] Y. Antolin, A. Minasyan, and A. Sisto. Commensurating endomorphisms of acylindrically hyperbolic groups and applications. *Groups Geom. Dyn.*, to appear, 2013.
- [Anc88] A. Ancona. Positive harmonic functions and hyperbolicity. In *Potential theory—surveys and problems (Prague, 1987)*, volume 1344 of *Lecture Notes in Math.*, pages 1–23. Springer, Berlin, 1988.
- [Anc90] A. Ancona. Théorie du potentiel sur les graphes et les variétés. In *École d’été de Probabilités de Saint-Flour XVIII—1988*, volume 1427 of *Lecture Notes in Math.*, pages 1–112. Springer, Berlin, 1990.

- [Ave72] A. Avez. Entropie des groupes de type fini. *C. R. Acad. Sci. Paris Sér. A-B*, 275:A1363–A1366, 1972.
- [Ave74] A. Avez. Théorème de Choquet-Deny pour les groupes à croissance non exponentielle. *C. R. Acad. Sci. Paris Sér. A*, 279:25–28, 1974.
- [BB07] S. Blachère and S. Brofferio. Internal diffusion limited aggregation on discrete groups having exponential growth. *Probab. Theory Related Fields*, 137(3-4):323–343, 2007.
- [BF02] M. Bestvina and K. Fujiwara. Bounded cohomology of subgroups of mapping class groups. *Geom. Topol.*, 6:6989 (electronic), 2002.
- [BH11] M. Björklund and T. Hartnick. Biharmonic functions on groups and limit theorems for quasimorphisms along random walks. *Geom. Topol.*, 15(1):123–143, 2011.
- [BHM08] S. Blachère, P. Haïssinsky, and P. Mathieu. Asymptotic entropy and Green speed for random walks on countable groups. *Ann. Probab.*, 36(3):1134–1152, 2008.
- [BHM11] S. Blachère, P. Haïssinsky, and P. Mathieu. Harmonic measures versus quasiconformal measures for hyperbolic groups. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(4):683–721, 2011.
- [Bow08] B. H. Bowditch. Tight geodesics in the curve complex. *Invent. Math.*, 171(2):281–300, 2008.
- [Bow12] B. H. Bowditch. Relatively hyperbolic groups. *Internat. J. Algebra Comput.*, 22(3):1250016, 66, 2012.
- [BP94] I. Benjamini and Y. Peres. Tree-indexed random walks on groups and first passage percolation. *Probab. Theory Related Fields*, 98(1):91–112, 1994.
- [BQa] Y. Benoist and J.-F. Quint. Central limit theorem for hyperbolic groups. *Preprint*.
- [BQb] Y. Benoist and J.-F. Quint. Central limit theorem for linear groups. *Annals of Probability, to appear*.
- [Bro03] J. F. Brock. The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores. *J. Amer. Math. Soc.*, 16(3):495–535 (electronic), 2003.
- [CM14] D. Calegari and J. Maher. Statistics and compression of scl. *Ergodic Theory Dynam. Systems, to appear*, 2014.
- [Der80] Y. Derriennic. Quelques applications du théorème ergodique sous-additif. In *Conference on Random Walks (Kleebach, 1979) (French)*, volume 74 of *Astérisque*, pages 183–201, 4. Soc. Math. France, Paris, 1980.
- [Der83] Y. Derriennic. Un théorème ergodique presque sous-additif. *Ann. Probab.*, 11(3):669–677, 1983.
- [DGO11] F. Dahmani, V. Guirardel, and D. Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. *arXiv:1111.7048*, 2011.
- [Dur] M. Durham. The augmented marking complex of a surface. *arXiv:1309.4065*.
- [EK13] A. G. Erschler and V. A. Kaĭmanovich. Continuity of asymptotic characteristics for random walks on hyperbolic groups. *Funktsional. Anal. i Prilozhen.*, 47(2):84–89, 2013.

- [Ers11] A. Erschler. On continuity of range, entropy and drift for random walks on groups. In *Random walks, boundaries and spectra*, volume 64 of *Progr. Probab.*, pages 55–64. Birkhäuser/Springer Basel AG, Basel, 2011.
- [FPS13] R. Frigerio, M. B. Pozzetti, and A. Sisto. Extending higher dimensional quasi-cocycles. *J. Topology*, to appear, 2013.
- [GL13] L. Gilch and F. Ledrappier. Regularity of the drift and entropy of random walks on groups. *Publ. Mat. Urug.*, 14:147–158, 2013.
- [GM08] D. Groves and J. F. Manning. Dehn filling in relatively hyperbolic groups. *Israel J. Math.*, 168:317–429, 2008.
- [Gou13] S. Gouezel. Martin boundary of measures with infinite support in hyperbolic groups. [arXiv:1302.5388](#), February 2013.
- [Gou15] S. Gouëzel. Analyticity of the entropy and the escape rate of random walks in hyperbolic groups. [arXiv:1509.06859](#), September 2015.
- [Ham62] J. M. Hammersley. Generalization of the fundamental theorem on sub-additive functions. *Proc. Cambridge Philos. Soc.*, 58:235–238, 1962.
- [Ham08] U. Hamenstädt. Bounded cohomology and isometry groups of hyperbolic spaces. *J. Eur. Math. Soc. (JEMS)*, 10(2):315349, 2008.
- [HMM13] P. Haissinsky, P. Mathieu, and S. Mueller. Renewal theory for random walks on surface groups. [arXiv:1304.7625](#), April 2013.
- [HO13] M. Hull and D. Osin. Induced quasicocycles on groups with hyperbolically embedded subgroups. *Algebr. Geom. Topol.*, 13(5):2635–2665, 2013.
- [Hor15] C. Horbez. Central limit theorems for mapping class groups and $\text{Out}(F_N)$. [arXiv:1506.07244](#), June 2015.
- [Kai00] V. A. Kaimanovich. The Poisson formula for groups with hyperbolic properties. *Ann. of Math. (2)*, 152(3):659–692, 2000.
- [KL07] A. Karlsson and F. Ledrappier. Linear drift and Poisson boundary for random walks. *Pure Appl. Math. Q.*, 3(4, Special Issue: In honor of Grigory Margulis. Part 1):1027–1036, 2007.
- [KV83] V. A. Kaĭmanovich and A. M. Vershik. Random walks on discrete groups: boundary and entropy. *Ann. Probab.*, 11(3):457–490, 1983.
- [Led12] F. Ledrappier. Analyticity of the entropy for some random walks. *Groups Geom. Dyn.*, 6(2):317–333, 2012.
- [Led13] F. Ledrappier. Regularity of the entropy for random walks on hyperbolic groups. *Ann. Probab.*, 41(5):3582–3605, 2013.
- [Mat14] P. Mathieu. Differentiating the entropy of random walks on hyperbolic groups. *Ann. Probab.*, to appear, 2014.
- [MM99] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999.

- [MM00] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. *Geom. Funct. Anal.*, 10(4):902–974, 2000.
- [MT14] J. Maher and G. Tiozzo. Random walks on weakly hyperbolic groups. [arXiv:1410.4173](#), 2014.
- [Osi14] D. Osin. Acylindrically hyperbolic groups. *Trans. Amer. Math. Soc.*, to appear, 2014.
- [Raf07] K. Rafi. A combinatorial model for the teichmüller metric. *GAFAGeometric And Functional Analysis*, 17(3):936–959, 2007.
- [Sis11] A. Sisto. Contracting elements and random walks. *J. Reine Angew. Math.*, to appear, 2011.
- [Sis14a] A. Sisto. Quasi-convexity of hyperbolically embedded subgroups. *Math. Z.*, to appear, 2014.
- [Sis14b] A. Sisto. Tracking rates of random walks. *Israel J. Math.*, to appear, 2014.
- [Ste86] J. M. Steele. An Efron-Stein inequality for nonsymmetric statistics. *Ann. Statist.*, 14(2):753–758, 1986.
- [Woe00] W. Woess. *Random walks on infinite graphs and groups*, volume 138 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2000.